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THE INVARIANCE PRINCIPLE FOR ASSOCIATED RANDOM FIELDS

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1. Introduction

Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be a random field on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. For $n \in N$ put

$$S_{n\underline{1}} = \sum_{\underline{1} \le \underline{j} \le n\underline{1}} X_{\underline{j}}, \tag{1.1}$$

assume

$$n^{-d} ES_{n\underline{1}}^2 \longrightarrow_n \sigma^2 \in (0, \infty), \tag{1.2}$$

and define

$$W_n(\underline{t}) = (\sigma n^{\frac{d}{2}})^{-1} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\underline{j}}, \tag{1.3}$$

where $W_n(\underline{t}) = 0$ for some $t_i = 0$. Then W_n is a measurable map from $(\Omega, \mathcal{F})$ into $(D_d, \mathcal{B}(D_d))$, where D_d is the set of all functions on $[0,1]^d$ which have left limits and are continuous from the right, and $\mathcal{B}(D_d)$ is the Borel σ -field induced by the Skorohod topology. $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ is said to fulfill the invariance principle if W_n converges weakly to the $d-$ parameter Wiener process W on D_d.

In this paper we investigate the invariance principle for random fields satisfying a condition of strong positive dependence called association. A finite collection $\{X_1, \cdots, X_m\}$ of random variables is associated if for any two coordinatewise nondecreasing functions f_1, f_2 on R^m such that $\hat{f}_i = f(X_1, \cdots, X_m)$ has finite variance for $i = 1, 2$, there holds $\mathrm{Cov}(\hat{f}_1, \hat{f}_2) \ge 0$. An infinite collection is associated if every finite

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subcollection is associated (cf. Esary, Proschan and Walkup [7]). Many recent papers have been concerned with limit theorems for associated sequences (see, for example, Newman [9]).

Burton and Waymire [5] extended the notion of association to the random measure and proved the central limit theorem for associated random measures.

Burton and Kim [4] obtained the following invariance principle for stationary associated random fields satisfying finite $\delta-$ susceptibility criterion which used a result of Bickel and Wichura [1] allowing them to conclude tightness.

THEOREM A (BURTON, KIM (1988)). *Let $\{X_{\underline{j}}\}$ be a stationary associated random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume there is a positive constant C so that for all $n \in N$*

$$E\left\{\left|\frac{S_{n\underline{1}}}{\sigma n^{\frac{d}{2}}}\right|^{2+\delta}\right\} \le C, \tag{1.4}$$

Then $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ fulfills the invariance principle.

Burton and Kim[4] applied Theorem A to the random measure as follows:

THEOREM B(BURTON, KIM(1988)). *Let X be a stationary associated random measure. If there is a constant $C < \infty$ depending only on X so that for all $A \supset I$ we have*

$$E[|X(A) - EX(A)|^{2+\delta}] \le C|A|^{1+\frac{\delta}{2}}.$$

where $|A|$ denotes the Lebesgue measure of A and $I - [0,1]^d$. Then X satisfies the invariance principle.

Birkel[3] extended the invariance principle of Newman and Wright [10] to nonstationary case and obtained the following invariance principle for one parameter associated processes

THEOREM C(BIRKEL(1988)). *Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0, EX_j^2 < \infty$ Assume*

$$E(W_n(s)W_n(t)) \longrightarrow_n min\{s,t\} \ \ for \ \ s,t \in [0,1] \tag{1.5}$$

$$\{(W_n(t) - W_n(s))^2 : n \in N, s, t \in [0,1]\} \text{ is uniformly integrable} \quad (1.6)$$

Then $\{X_j; j \in N\}$ *fulfills the invariance principle.*

Kim and Han[8] improved the invariance principle of Birkel[3] to a two-parameter case by applying Theorem 10 of Newman and Wright [10]. The problem of extension of this invariance principle to the $d-$parameter case ($d > 2$) is still an open problem[10].

Our aim of this paper is to extend Theorems A and B to the nonstationary case by adding a condition on the covariance structure and to provide a new invariance principle for an array of nonstationary associated multiparameter random variables by strengthening the hypothesis of uniform integrability of Theorem C.

In Section 2 we introduce some preliminary results for the proof of the invariance principle for nonstationary associated random fields. In Section 3 we will obtain a general invariance principle for $d-$ parameter associated processes (Theorem 3.1) which requires no stationarity by combining the ideas of Theorems A and C and apply this notion to the associated random measure in Section 4.

2. Some results for associated random fields

If $\underline{t} = (t_1, t_2, \cdots, t_d)$, let $|\underline{t}|$ stand for the product $t_1 t_2 \cdots t_d$, and $\| \underline{t} \| = \max(|t_1|, |t_2|, \cdots, |t_d|)$.

THEOREM 2.1. *Let* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *be an associated random field with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ *and define* $W_n(\cdot)$ *as in (1.3). Assume*

$$E\{W_n^2(\underline{t})\} \longrightarrow_n |\underline{t}| \text{ for } \underline{0} \leq \underline{t} \leq \underline{1}. \quad (2.1)$$

Then the following conditions are equivalent:

(i) $E\{W_n(\underline{s})W_n(\underline{t})\} \longrightarrow_n |\underline{s}|$ *for* $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{1}$,

(ii) $E\{W_n(\underline{1})W_n(\underline{t})\} \longrightarrow_n |\underline{t}|$ *for* $\underline{0} \leq \underline{t} \leq \underline{1}$,

(iii) $E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \longrightarrow_n 0$, *for* $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$.

Proof. (i) $\Rightarrow$ (ii). (ii) follows from (i) by taking $\underline{s} = \underline{1}$.

(ii) $\Rightarrow$ (iii). Since the random variables are nonnegatively correlated, it follows from (2.1) and (ii) of Theorem 2.1 that

$$\begin{aligned} 0 &\le E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \\ &\le E\{(W_n(\underline{t}) - W_n(\underline{0}))(W_n(\underline{1}) - W_n(\underline{t}))\} \\ &= E\{W_n(\underline{t})W_n(\underline{1})\} - E\{W_n^2(\underline{t})\} \longrightarrow_n 0 \end{aligned}$$

(iii) $\Rightarrow$ (i).
$E\{W_n(\underline{s})W_n(\underline{t})\}$
$= E\{(W_n(\underline{s}) - W_n(\underline{0}))(W_n(\underline{t}) - W_n(\underline{s}) + W_n(\underline{s}) - W_n(\underline{0}))\}$
$= E\{(W_n(\underline{s}) - W_n(\underline{0}))(W_n(\underline{t}) - W_n(\underline{s})) + E(W_n(\underline{s}) - W_n(\underline{0}))^2\} \longrightarrow_n |\underline{s}|$
according to (2.1) and (iii) of Theorem 2.1.

THEOREM 2.2. *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be an associated random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ and define $W_n(\cdot)$ as in (1.3). If $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ fulfills the invariance principle, then*

$$E(W_n(\underline{s})W_n(\underline{t})) \longrightarrow_n |\underline{s}| \text{ for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}. \tag{2.2}$$

Proof. Since the invariance principle is fulfilled, $\{W_n^2(\underline{t}) : n \in N\}$ is uniformly integrable and hence

$$\mathrm{E}\{W_n^2(\underline{t})\} \longrightarrow_n E\{W^2(\underline{t})\} = |\underline{t}| \text{ for } 0 \le \underline{t} \le \underline{1},$$

according to Theorem 5.4 of Billingsley[2]. By Theorem 2.1 it remains to prove

$$E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \longrightarrow_n 0 \tag{2.3}$$

for $\underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v} \le \underline{1}$.
To prove (2.3) : Let $\underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v} \le \underline{1}$ be given. Since the invariance principle is fulfilled, $\{W_n^2(t) : n \in N\}$ is uniformly integrable. Hence

$$\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u})) : n \in N\} \tag{2.4}$$

is uniformly integrable. According to Theorem 5.4 of Billingsley[3] and (2.4)

$$E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\}$$

$$\longrightarrow_n E\{(W(\underline{t}) - W(\underline{s}))(W(\underline{v}) - W(\underline{u}))\}.$$

But

$$E\{(W(\underline{t}) - W(\underline{s}))(W(\underline{v}) - W(\underline{u}))\}$$

$$= E\{W(\underline{t}) - W(\underline{s})\}E\{W(\underline{v}) - W(\underline{u})\} = 0$$

which proves (2.3).

Theorem 2.2 shows that (2.2) is a weak form of stationarity and a necessary condition for the invariance principle.

3. An invariance principle

A subset B of $[0,1]^d$ is called a block if it is of the form $\Pi_1^d(s_j, t_j]$, where the $(s_j, t_j]'s, j = 1, \cdots, d$, are half closed subintervals of $[0,1]$. For each $i, 1 \leq i \leq d$, let

$$0 < a_1^{(i)} < b_1^{(i)} < a_2^{(i)} < b_2^{(i)} < \cdots < a_n^{(i)} < b_n^{(i)} = 1$$

be real numbers. Call a collection of blocks in $[0,1]^d$ "strongly separated" if it is of the form $\{\Pi_1^d(a_k^{(i)}, b_k^{(i)}] : 1 \leq k \leq n, 1 \leq i \leq d\}$ or if it is a subfamily of such a family of blocks.

Disjoint blocks B and F are neighboring if they abut (for example, when $d = 3$ the blocks $(s,t] \times (a,b] \times (c,d]$ and $(t,u] \times (a,b] \times (c,d]$ are neighboring $(0 \leq s < t < u \leq 1)$. For each block $B = (\underline{s}, \underline{t}] = \Pi_1^d(s_i, t_i]$, let

$$S_n(B) = \sum_{\underline{j} \in nB} X_{\underline{j}}, \quad W_n(B) = (\sigma n^{\frac{d}{2}})^{-1} S_n(B) \tag{3.1}$$

where $nB = (n\underline{s}, n\underline{t}] = \Pi_1^d(ns_i, nt_i]$ for $B = (\underline{s}, \underline{t}]$. If we consider $X = \{X(\underline{t}) : \underline{t} \in [0,1]^d\}$ as a stochastic process, then the increment $X(B)$ of X around a block $B = \Pi_1^d(s_j, t_j]$ is given by

$$X(B) = \sum_{\varepsilon_1 = 0,1} \cdots \sum_{\varepsilon_d = 0,1} (-1)^{d - \sum \varepsilon_j}$$
$$X(s_1 + \varepsilon_1(t_1 - s_1), s_2 + \varepsilon_2(t_2 - s_2), \cdots, s_d + \varepsilon_d(t_d - s_d)).$$

THEOREM 3.1. *Let* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *be an associated random field with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ *and define* $W_n(\cdot)$ *as in (1.3). Assume*

$$E\{W_n(\underline{s})W_n(\underline{t})\} \longrightarrow_n |\underline{s}| \text{ for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}, \tag{3.2}$$

$$E|W_n(B)|^{2+\delta} \le C|B|^{1+\frac{\delta}{2}}, \tag{3.3}$$

where $B = (\underline{s}, \underline{t}]$ *for* $\underline{0} \le \underline{s} \le \underline{t} \le \underline{1}$.

Then $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *fulfills the invariance principle.*

Proof. By Lemma 2 of Deo(1975) it is sufficient to show that $W_n(\cdot)$ converges weakly in the Skorokhod topology to a stochastic process W which has the following properties:

(a) $E\{W(\underline{t})\} = 0, E\{W(\underline{t})^2\} = |\underline{t}|, \quad \underline{0} < \underline{t} \le \underline{1},$

(b) W has continuous paths,

(c) Increments of W around any collection of strongly separated blocks in $[0,1]^d$ are independent random variables.

Note that for a block $B = \Pi_1^d(s_i, t_i] \subset [0,1]^d$

$$W_n(B) = (\sigma n^{\frac{d}{2}})^{-1} \sum_{\underline{j} \in nB} X_{\underline{j}} \tag{3.4}$$

where $nB = \Pi_1^d(ns_i, nt_i]$.

From Chebyshev's inequality, Schwarz inequalty, and (3.3) it follows that for neighboring blocks B and F

$$\begin{aligned} &P[\min(|W_n(B)|, |W_n(F)|) \ge \lambda] \\ &\le \lambda^{-(2+\delta)} E[\{\min(|W_n(B)|, |W_n(F)|)\}^{2+\delta}] \\ &\le \lambda^{-(2+\delta)} [E\{|W_n(B)|^{2+\delta}\} E\{|W_n(F)|^{2+\delta}\}]^{\frac{1}{2}} \\ &\le \lambda^{-(2+\delta)} [C(|B|)^{1+\frac{\delta}{2}} C(|F|)^{1+\frac{\delta}{2}}]^{\frac{1}{2}} \\ &\le \lambda^{-(2+\delta)} C[(|B||F|)^{\frac{1}{2}}]^{1+\frac{\delta}{2}} \\ &\le \lambda^{-(2+\delta)} C(|B| + |F|)^{1+\frac{\delta}{2}} \\ &= \lambda^{-(2+\delta)} C|B \cup F|^{1+\frac{\delta}{2}}. \end{aligned} \tag{3.5}$$

Thus by Theorem 3 of Bickel and Wichura(1971) the following tightness condition (3.6) is in force

$$\lim_{n \in N} \sup P\{w(W_n, \delta) > \varepsilon\} \to 0 \text{ as } \delta \downarrow 0 \tag{3.6}$$

where $w(W_n,\delta) = \sup_{\|\underline{s}-\underline{t}\|<\delta} |W_n(\underline{s}) - W_n(\underline{t})|$ and $\| \underline{s} - \underline{t} \| = \max\{|s_1 - t_1|, \cdots, |s_d - t_d|\}$ and thus the sequence $\{W_n\}$ is tight.
It should be noted that Bickel and Wichura[1] assumed that $W_n(\cdot)$ vanishes along lower boundary of $[0,1]^d$:

$$\sum_{1\le p\le d} [0,1] \times \cdots \times [0,1] \times \{0\} \times [0,1] \times \cdots \times [0,1]$$

($\{0\}$ is in the pth position). But by (3.3), $P(\sum_{\underline{j}\in B} X_{\underline{j}} = 0) = 1$ if $|B| = 0$, so a version of W_n exists which is zero along the lower boundary. Let X be a limit in distribution of a subsequence of $\{W_n : n \in N\}$. Then it follows from (3.6) and Theorem 15.5 of Billingsley[2] that X is continuous with probability one. It suffices to show that X is distributed like W. From assumption it is easily seen that

$$EW_n(\underline{t}) \to_n 0, EW_n^2(\underline{t}) \to_n |\underline{t}|. \tag{3.7}$$

By (3.3) for n large enough

$$E(|W_n(\underline{t})|^{2+\delta}) \le C|\underline{t}|^{1+\frac{\delta}{2}} \tag{3.8}$$

and so $\{W_n^2(\underline{t}) : n \in N\}$ and $\{W_n(\underline{t}) : n \in N\}$ are uniformly integrable for every $\underline{t} \in [0,1]^d$. As

$$W_n(\underline{t}) \to_n X(\underline{t}), \quad W_n^2(\underline{t}) \to_n X^2(\underline{t})$$

in distribution (for a subsequence), Theorem 5.4 of Billingsley[2] and (3.7) imply

$$EX(\underline{t}) = 0, \quad EX^2(\underline{t}) = |\underline{t}|.$$

According to Theorem 19.1 of Billingsley[2], X is distributed like W if X has independent increments, that is for the strongly separated blocks, $B_1, B_2, \cdots, B_k$,

$$X(B_1), X(B_2), \cdots, X(B_k) \text{ are independent for all } k \in N, \tag{3.9}$$

where $B_k = (\underline{t}_{k-1}, \underline{t}_k]$, $\underline{0} \le \underline{t}_0 \le \cdots \le \underline{t}_k \le \underline{1}$

To show (3.9): Since

$$(W_n(B_1),\cdots,W_n(B_k)) \to_n (X(B_1),\cdots,X(B_k))$$

in distribution, and since the $W_n(B_i)'s$ are associated by (P_4) of Esary, Proschan and Walkup [7] $X(\underline{t}_1)-X(\underline{t}_0),\cdots,X(\underline{t}_k)-X(\underline{t}_{k-1})$ are associated, according to (P_5) of [7]. A similar argument as above (using Theorem 5.4 of Billingsley [2] and the fact that associated random variables are nonnegatively correlated) yields, for $i \neq j, B_i = (\underline{s},\underline{t}]$ and $B_j = (\underline{u},\underline{v}]$,

$$\begin{aligned}
\mathrm{Cov}(X(B_i),X(B_j)) &= \lim_{n\to\infty}\mathrm{Cov}(W_n(B_i),W_n(B_j)) \\
&= \lim_{n\to\infty}\mathrm{Cov}(W_n((\underline{s},\underline{t}]),W_n((\underline{u},\underline{v}])) \\
&\le \lim_{n\to\infty}\mathrm{Cov}(W_n(\underline{t})-W_n(\underline{s}),W_n(\underline{v})-W_n(\underline{u})) \\
&= 0, \quad \underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v} \le \underline{1},
\end{aligned}$$

according to (iii) of Theorem 2.1. Hence the $X(B_i)'s$ are associated and uncorrelated random variables and thus independent by Corollary 3 of Newman[9]. This proves (3.9) and therefore the proof of Theorem 3.1 is complete.

4. Applications

In this section we will apply the notions of associated random fields to the random measures, that is, a simple argument using Chebyshev's inequality allows us to extend the invariance principle for associated random fields to random measure. $\mathcal{B}^d$ denotes the collection of Borel subsets of d--dimensional Euclidean space R^d. The space M of all nonnegative measure μ defined on $(R^d,\mathcal{B}^d)$ and finite on bounded sets will be equipped with the smallest σ-field containing basic sets of the form $\{\mu \in M : \mu(A) \le r\}$ for $A \in \mathcal{B}^d$, $0 \le r < \infty$. A random measure X is a measurable map from a probability space $(\Omega,\mathcal{F},P)$ into $(M,\mathcal{M})$, the induced measure $P_X = P \circ X^{-1}$ on $(M,\mathcal{M})$ is the distribution of X and if X is a random measure and $\mathcal{B}^d$ is a Borel subset of R^d then $X(B)$ represent the random mass of the region B.

For the random measure X define the K-renormalization of X to be the signed random measure X_K where

$$X_K(B) = \frac{X(KB)-EX(KB)}{\sigma K^{\frac{d}{2}}} \tag{4.1}$$

and let $X_K(\underline{t}) = X_K(t_1, \cdots, t_d)$ be defined by

$$X_K(\underline{t}) = X_K((0, t_1] \times \cdots \times (0, t_d]) \tag{4.2}$$

for $\underline{t} \in [0, \infty)^d$. Let $\{X_K\}$ be a sequence of random measures on R^d. A set function X_K satisfies the central limit theorem if for any bounded $B \in \mathcal{B}^d, X_K(B)$ converges in distribution to $N(0, |B|)$ as $K \to \infty$ where $X_K(B)$ is difined in (4.1) and $|B|$ denotes the Lebesgue measure of B and the random measure X satisfies the invariance principle if X_K converges weakly to the d–dimensional Wiener measure W.

DEFINITION 4.1 (BURTON,WAYMIRE(1985)). A random measure X is associated if and only if the family of random variables $\mathcal{F} = \{X(B) :\ \text{B a Borel set}\ \}$ is associated.

THEOREM 4.2. *Let X be an associated random measure with $EX(B) = 0$, $EX^2(B) < \infty$ and define $X_K(\underline{t})$ as in (4.2). Assume*

$$E\{X_K(\underline{s})X_K(\underline{t})\} \to_K |\underline{s}| \quad \textit{for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}. \tag{4.3}$$

For $A \in \mathcal{B}^d, A$ bounded, $|A| > 1$, there exists constants $C' < \infty$, and $\delta > 0$ such that

$$E(|X(A) - EX(A)|^{2+\delta}) \le C'(\sigma^2|A|)^{(1+\frac{\delta}{2})}. \tag{4.4}$$

Then X satisfies the invariance principle.

Proof. Note that for a block $B \subset [0, 1]^d$

$$X_K(B) = \frac{X(KB) - EX(KB)}{\sigma K^{\frac{d}{2}}} \tag{4.5}$$

where, if $B = \Pi_{i=1}^d (s_i, t_i]$, then $KB = \Pi_{i=1}^d (Ks_i, Kt_i]$. Like in (3.5) from (4.4), it follows that for neighboring blocks B and F,

$$\begin{aligned} &P[min(|X_K(B)|, |X_K(F)|) \ge \lambda] \\ &\le \lambda^{-(2+\delta)} C'(|B \cup F|)^{1+\frac{\delta}{2}} \end{aligned}$$

and thus by Theorem 3 of Bickel and Wichura[1] the sequence $\{X_K\}$ is tight. Like in the proof of Theorem 3.1, by (4.4), $P(X(A) = 0) = 1$ if $|A| = 0$, so a version of X_K exists which is 0 along the lower boundary.

Suppose X is the limit in distribution of a subsequence. Then X is continuous with probability one by the similar arguments in the proof of Theorem 3.1. It suffices to show that X is distributed as W. From (4.5) and condition (4.3) it is easily seen

$$E(X_K(\underline{t})) = 0, \quad EX_K^2(\underline{t}) \to_K |\underline{t}|. \tag{4.6}$$

By condition (4.4), for K large enough,

$$E(|X_K(\underline{t})|^{2+\delta}) \leq \frac{1}{(\sigma K^{\frac{d}{2}})^{2+\delta}} C'(\sigma^2 K^d |\underline{t}|)^{1+\frac{\delta}{2}}$$

and so $\{X_K(\underline{t})\}$ and $\{X_K^2(\underline{t})\}$ are uniformly integrable for every $\underline{t} \in [0,1]^d$. As

$$X_K(\underline{t}) \to_K X(\underline{t}), \ \ X_K^2(t) \to_K X^2(t)$$

in distribution, Theorem 5.4 of Billingsley[3] and (4.6) imply that

$$EX(\underline{t}) = 0, \quad EX^2(\underline{t}) = |\underline{t}|.$$

Finally let $B_1, \cdots, B_m \subset [0,1]^d$ be strongly separated blocks, and let $B_i = (\underline{s}, \underline{t}], B_j = (\underline{u}, \underline{v}]$, where $\underline{0} \leq s \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$. Since random variables $X(I_{\underline{j}})$ are nonnegative correlated it follows from (4.3) that

$$\begin{aligned} &\mathrm{Cov}(X_K(B_i), X_K(B_j)) \\ \leq &\mathrm{Cov}(X_K(\underline{t}) - X_K(\underline{s}), X_K(\underline{v}) - X_K(\underline{u})) \to_K 0 \end{aligned} \tag{4.7}$$

according to Theorem 2.1, where $I_{\underline{j}} = (\underline{j} - \underline{1}, \underline{j}]$ for $\underline{1} \leq \underline{j} \in Z^d$. Since $X_K(B_{\underline{j}})'s$ are associated, by Corollary 3 of Newman[9] and (4.7) the $X_K(B_{\underline{j}})'s$ are independent as $K \to \infty$. Hence X must have independent increments. Thus, every subsequence $\{X_{K'}\}$ of $\{X_K\}$ has further subsequence of $\{X_{K''}\}$ which converge weakly to the Wiener measure W on $[0,1]^d$. It follows that X_K converges weakly to the d-dimensional Wiener measure W.

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