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COUNTER-EXAMPLES AND DUAL OPERATOR ALGEBRAS WITH PROPERTIES $(\mathbb{A}_{m,n})$

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology coincides with the weak* topology on $\mathcal{L}(\mathcal{H})$ (cf. [6]). Several functional analysists have studied the problem of solving systems of simultaneous equations in the predual of a dual algebra (cf. [3]). This theory is applied to the study of invariant subspaces and dilation theory, which are deeply related to the classes $\mathbb{A}_{m,n}$ (that will be defined below) (cf. [3]). An abstract geometric criterion for dual algebras with property $(\mathbb{A}_{\aleph_0, \aleph_0})$ was first given in [1]. In particular, properties $X_{\theta,\gamma}$ and $E^r_{\theta,\gamma}$ (or $E^\ell_{\theta,\gamma}$), $0 \le \theta < \gamma$, have been studied as geometric criteria for membership of certain classes $\mathbb{A}_{\aleph_0,\aleph_0}$ and $\mathbb{A}_{1,\aleph_0}$ (or $\mathbb{A}_{\aleph_0,1}$) respectively (cf. [1],[4],[5], and [7]). We consider the following question:

QUESTION 1.1. *Does a dual algebra $\mathcal{A}$ have property $(\mathbb{A}_{1,\aleph_0})$ if $\mathcal{A}$ has property $E^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma$?*

This question has been motivated from the result in [3] that if a dual algebra $\mathcal{A}$ has property $X_{\theta,\gamma}$ for some $0 \le \theta < \gamma$, then $\mathcal{A}$ has property $(\mathbb{A}_{\aleph_0,\aleph_0})$. Before we start the work, we recall some definitions and terminology concerning the theory of dual algebras (cf. [3],[4]). The notation employed herein agrees with that in [3] and [12].

Let $C_1(\mathcal{H})$ be the Banach space of trace class operators on $\mathcal{H}$ equipped with the trace norm. If $\mathcal{A}$ is a dual algebra, then it follows from [3]

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that $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing $< T, [L]_{\mathcal{A}} >= trace(TL)$, $T \in \mathcal{A}$, $[L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}$. The Banach space $\mathcal{Q}_{\mathcal{A}}$ is called a ***predual space*** of $\mathcal{A}$. We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. For x and y in $\mathcal{H}$, we define $(x \otimes y)(u) = (u, y)x$, for all $x \in \mathcal{H}$.

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{A}_T$ the dual algebra generated by T and denote by $\mathcal{Q}_T$ the predual space $\mathcal{Q}_{\mathcal{A}_T}$ of $\mathcal{A}_T$.

Suppose m and n are cardinal numbers such that $1 \le m, n \le \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \le i < m$, $0 \le j < n$, where $\{[L_{ij}]\}_{\substack{0\le i<m\\ 0\le j<n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0\le i<m}$, $\{y_j\}_{0\le j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$.

We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the boundary of $\mathbb{D}$. The space $L^p = L^p(\mathbb{T})$, $1 \le p \le \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\mathbb{T}$. The space $H^p = H^p(\mathbb{T}), 1 \le p \le \infty$, is the usual Hardy space. It is well-known (cf. [9]) that the space H^∞ is the dual space of L^1/H_0^1, where

$$H_0^1 = \left\{ f \in L^1 : \int_0^{2\pi} f(e^{it}) e^{int}\, dt = 0, \text{ for } n = 0, 1, 2, \dots \right\} \qquad (1.1)$$

and the duality is given by the pairing

$$\langle f, [g] \rangle = \int_{\mathbb{T}} fg\, dm, \quad \text{for } f \in H^\infty,\ [g] \in L^1/H_0^1. \qquad (1.2)$$

A contraction operator $T \in \mathcal{L}(\mathcal{H})$ is ***absolutely continuous*** if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0).

Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then according to [3, Theorem 4.1] there exists a functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ for every f in H^∞. The mapping Φ_T is a norm-decreasing, weak* continuous algebra homomorphism, and the range of Φ_T is weak* dense in $\mathcal{A}_T$. Furthermore, there exists a bounded, linear, one-to-one map ϕ_T of $\mathcal{Q}_T$ into L^1/H_0^1 such that

$\Phi_T = \phi_T^*$. The mapping Φ_T is said to be *Foiaş Nagy functional calculus*. We define by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if m and n are any cardinal numbers such that $1 \le m, n \le \aleph_0$, we define by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$.

Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and θ is a nonnegative real number. We denote by $\mathcal{X}_\theta(\mathcal{A})$ the set of all $[L]$ in $\mathcal{Q}_\mathcal{A}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from the closed unit ball of $\mathcal{H}$ satisfying $\overline{\lim}_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \le \theta$ and $\|[x_i \otimes z]\| + \|[z \otimes y_i]\| \to 0$, for all z in $\mathcal{H}$. For $0 \le \theta < \gamma$, the dual algebra $\mathcal{A}$ is said to have property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathcal{X}_\theta(\mathcal{A})$ (i.e. the smallest closed convex and balanced set containing $\mathcal{X}_\theta(\mathcal{A})$) contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $\mathcal{Q}_\mathcal{A}$: $\overline{\text{aco}}(\mathcal{X}_\theta(\mathcal{A})) \supset \{[L] \in \mathcal{Q}_\mathcal{A} : \|[L]\| \le \gamma\} = B_{0,\gamma}$.

The following is a geometric criterion for property $(\mathbb{A}_{\aleph_0})$.

THEOREM 1.2 [3, Theorem 3.7]. *If a dual algebra $\mathcal{A}$ has property $X_{\theta,\gamma}$ for some $0 \le \theta < \gamma$, then $\mathcal{A}$ has property $(\mathbb{A}_{\aleph_0})$. In particular, if $T \in \mathbb{A}$, then $\mathcal{A}_T$ has property $X_{\theta,\gamma}$ for some $0 \le \theta < \gamma$ if and only if $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$.*

Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $0 \le \theta < \gamma \le 1$. We denote by $\mathcal{E}_\theta^r(\mathcal{A})$ ($\mathcal{E}_\theta^\ell(\mathcal{A})$ resp.) the set of all $[L]$ in $\mathcal{Q}_\mathcal{A}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ from the closed unit ball of $\mathcal{H}$ satisfying $\overline{\lim}_{i\to\infty} \|[L] - [x_i \otimes y_i]\| \le \theta$ and $\|[x_i \otimes z]\| \to 0$, for all $z \in \mathcal{H}$, ($\|[z \otimes y_i]\| \to 0$, for all $z \in \mathcal{H}$ resp.). A dual algebra $\mathcal{A}$ is said to have property $E_{\theta,\gamma}^r$ ($E_{\theta,\gamma}^\ell$ resp.), for some $0 \le \theta < \gamma \le 1$, if $\overline{\text{aco}}(\mathcal{E}_\theta^r(\mathcal{A})) \supset B_{0,\gamma}$ ($\overline{\text{aco}}(\mathcal{E}_\theta^\ell(\mathcal{A})) \supset B_{0,\gamma}$ resp.).

The following is also a geometric criterion for property $(\mathbb{A}_{1,\aleph_0})$ or $(\mathbb{A}_{\aleph_0,1})$.

THEOREM 1.3 [4, THEOREM 6.2]. *If $T \in \mathbb{A}$, then $\mathcal{A}_T$ has property $(\mathbb{A}_{1,\aleph_0})$ (or $(\mathbb{A}_{\aleph_0,1})$, resp.) if and only if $\mathcal{A}_T$ has property $E_{\theta,\gamma}^r$ (or $E_{\theta,\gamma}^\ell$, resp.) for some $0 \le \theta < \gamma \le 1$.*

Hence according to Theorem 1.2, Theorem 1.3, and definitions of properties $X_{\theta,\gamma}$ and $E_{\theta,\gamma}^r$ it is natural to give Question 1.1, which could be expected to be affirmative. But we obtain some counter-examples for the question in section 2. In section 3 we study matrices of dual algebras with properties $(\mathbb{A}_{m,n})$ and further examples.

2. Counter-examples for a geometric criterion

Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We write $\mathcal{M}_n(\mathcal{A})$ for the subalgebra of $\mathcal{L}(\mathcal{H}^{(n)})$ consisting of all $n \times n$ matrices with entries from $\mathcal{A}$, where $\mathcal{H}^{(n)} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{(n)}$. Then it follows from [1, Proposition 1.2] that $\mathcal{M}_n(\mathcal{A})$ is a dual algebra. In particular, the predual space $\mathcal{Q}_{\mathcal{M}_n(\mathcal{A})}$ is identified with the Banach space $\mathcal{M}_n(\mathcal{Q}_{\mathcal{A}})$ consisting of all $n \times n$ matrices with entries from $\mathcal{Q}_{\mathcal{A}}$. The duality is given by the pairing

$$\langle (T_{ij}), ([L_{ij}]) \rangle = \sum_{i,j=1}^{n} \langle T_{ij}, [L_{ij}] \rangle, \tag{2.1}$$

$(T_{ij}) \in \mathcal{M}_n(\mathcal{A})$, $([L_{ij}]) \in \mathcal{M}_n(\mathcal{Q}_{\mathcal{A}})$. If $\widetilde{x} = (x_1, \cdots, x_n)$ and $\widetilde{y} = (y_1, \cdots, y_n)$ belong to $\mathcal{H}^{(n)}$, then $[\widetilde{x} \otimes \widetilde{y}]_{\mathcal{M}_n(\mathcal{A})}$ is identified with the $n \times n$ matrix $([x_j \otimes y_i]_{\mathcal{A}})$. It follows from [1, Proposition 1.3] that if $\mathcal{A}$ is a dual algebra and n is a positive integer, then $\mathcal{A}$ has property $(\mathbb{A}_n)$ if and only if $\mathcal{M}_n(\mathcal{A})$ has property $(\mathbb{A}_1)$. This fact will be improved in section 3 (cf. Proposition 3.1).

The following lemma is a tool for this work.

LEMMA 2.1. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra that has property $E^r_{0,\gamma}$ ($E^\ell_{0,\gamma}$ resp.) for some real number $\gamma > 0$. Then for each positive integer n, the dual algebra $\mathcal{M}_n(\mathcal{A})$ has property $E^r_{0,\gamma/n^2}$ ($E^\ell_{0,\gamma/n^2}$ resp.).*

Proof. The idea of this proof comes from that of [1, Proposition 1.6]. We sketch the proof here. We set $\mathcal{B} = \mathcal{M}_n(\mathcal{A})$ for a simple notation. Then by [3, Proposition 1.21], it is sufficient to show that

$$\sup_{([L_{ij}]) \in \mathcal{E}^r_0(\mathcal{B})} |\langle (A_{ij}), ([L_{ij}]) \rangle| \geq (\gamma/n^2) \|(A_{ij})\| \tag{2.2}$$

for every matrix (A_{ij}) in $\mathcal{M}_n(\mathcal{A})$. To do so, letting $[L] \in \mathcal{E}^r_0(\mathcal{A})$, there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from the closed unit ball of $\mathcal{H}$ such that $\overline{\lim}\,\|[L] - [x_i \otimes y_i]\| = 0$ and $\|[x_i \otimes z]\| \longrightarrow 0$ for all $z \in \mathcal{H}$. For any fixed i_0 and j_0, $1 \leq i_0,\ j_0 \leq n$, and any positive integer i, we define

$$\tilde{x}_{i,i_0} = (\underbrace{\overbrace{0, \cdots, 0, x_i}^{(i_0)}, 0, \cdots, 0}_{(n)}) \tag{2.3a}$$

and

$$\tilde{y}_{i,j_0} = (\underbrace{\overbrace{0, \cdots, 0, x_i}^{(j_0)}, 0, \cdots, 0}_{(n)}). \tag{2.3b}$$

Then it is easy to show that $\|\tilde{x}_{i,i_0}\| \leq 1$, $\|\tilde{y}_{i,j_0}\| \leq 1$ for all $i \in \mathbb{N}$ and $\|[\tilde{x}_{i,i_0} \otimes \tilde{z}]\| \longrightarrow 0$ for all $\tilde{z} \in \mathcal{H}^{(n)}$. Finally, if we follow the proof of [1, Proposition 1.6], we can prove (2.2). Hence the proof is complete. □

LEMMA 2.2. *If a dual algebra* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *has property* $E^r_{0,\gamma-\theta}$ ($E^\ell_{0,\gamma-\theta}$ *resp.) for* $0 \leq \theta < \gamma$, *then* $\mathcal{A}$ *has property* $E^r_{\theta,\gamma}$ ($E^\ell_{\theta,\gamma}$ *resp.).*

Proof. Assume that $\mathcal{A}$ has property $E^r_{0,\gamma-\theta}$. Let $[\Lambda]$ be a coset in $\mathcal{Q}_{\mathcal{A}}$ with $\|[\Lambda]\| \leq \gamma$. Then it is sufficient to show that $[\Lambda] \in \overline{\text{aco}}\mathcal{E}^r_\theta(\mathcal{A})$. Since

$$\|((\gamma - \theta)/\gamma)[\Lambda]\| \leq \gamma - \theta, \tag{2.4}$$

we have $(\gamma - \theta)/\gamma)[\Lambda] \in \overline{\text{aco}}\mathcal{E}^r_0(\mathcal{A})$. Given $\epsilon > 0$, there exist a set $\{\alpha_k\}_{k=1}^n$ of complex numbers and $\{[L_k]\}_{k=1}^n \subset \mathcal{E}^r_0(\mathcal{A})$ such that $\alpha_k \geq 0$, $\sum_{k=1}^n \alpha_k = 1$, and

$$\left\|\frac{\gamma - \theta}{\gamma}[\Lambda] - \sum_{k=1}^n \alpha_k[L_k]\right\| < \epsilon. \tag{2.5}$$

So we have

$$\left\|[\Lambda] - \sum_{k=1}^n \alpha_k([L_k] + \frac{\theta}{\gamma}[\Lambda])\right\| < \epsilon. \tag{2.6}$$

Moreover, since $[L_k] \in \mathcal{E}^r_0(\mathcal{A})$ and $\|(\theta/\gamma)[\Lambda]\| \leq \theta$, we have $[L_k] + (\theta/\gamma)[\Lambda] \in \mathcal{E}^r_\theta(\mathcal{A})$ for $1 \leq k \leq n$. Thus $[\Lambda] \in \overline{\text{aco}}\mathcal{E}^r_\theta(\mathcal{A})$ and the proof is complete. □

According to Theorem 1.3, Lemma 2.1 and Lemma 2.2, we obtain easily the following theorem.

THEOREM 2.3. *Suppose $n \in \mathbb{N}$ and $0 \leq \theta < \gamma \leq 1$ with $\gamma - \theta \leq 1/n^2$. If $T \in \mathbb{A}_{1,\aleph_0}$ (or $\mathbb{A}_{\aleph_0,1}$), then $\mathcal{M}_n(\mathcal{A}_T)$ has property $E^r_{\theta,\gamma}$ (or $E^\ell_{\theta,\gamma}$, resp.).*

Recall (cf. [10, Corollary 4.8]) that if S is the unilateral shift of multiplicity one, then $S^{(n)} = \underbrace{S \oplus \cdots \oplus S}_{(n)} \in \mathbb{A}_{n,\aleph_0} \setminus \mathbb{A}_{n+1,1}$.

The following corollary gives a negative answer for Question 1.1.

COROLLARY 2.4. *If S is a unilateral shift operator of multiplicity one, then the dual algebra $\mathcal{M}_n(\mathcal{A}_S)$ has property $E^r_{0,1/n^2}$ but not property $(\mathbb{A}_1)$ for any positive integer $n \geq 2$.*

Proof. Since $S \in \mathbb{A}_{1,\aleph_0}$, it follows from Theorem 1.3 that $\mathcal{A}_S$ has property $E^r_{0,1}$. By Lemma 2.1, the dual algebra $\mathcal{M}_n(\mathcal{A}_S)$ has property $E^r_{0,1/n^2}$. Now suppose that the dual algebra $\mathcal{M}_n(\mathcal{A}_S)$ has property $(\mathbb{A}_1)$. By [2, Proposition 2.3], $\mathcal{A}_S$ has property $(\mathbb{A}_n)$ and $S \in \mathbb{A}_n$, which contradicts the above remark. Hence the proof is complete. □

Recall from [8] that if U is a bilateral shift of multiplicity one, then we have $U^{(n)} \in \mathbb{A}_n \backslash \mathbb{A}_{n+1}$, which implies that $S^{(n)} \oplus S^* \notin \mathbb{A}_{n+2}$, $n \in \mathbb{N}$.

COROLLARY 2.5. *Let S be the unilateral shift operator of multiplicity one. Suppose $0 \leq \theta < \gamma \leq 1$ and $\gamma - \theta \leq 1/(n+2)^2$ for some $n \in \mathbb{N}$. Then the dual algebra $\mathcal{M}_{n+2}(\mathcal{A}_{S^{(n)} \oplus S^*})$ has property $E^r_{\theta,\gamma}$ and property $E^\ell_{\theta,\gamma}$, but not property $(\mathbb{A}_1)$.*

Proof. Apply the proof of Corollary 2.4 and the fact that $S^{(n)} \oplus S^* \in \mathbb{A}_{1,\aleph_0} \cap \mathbb{A}_{\aleph_0,1}$ for the first part. Moreover, the fact that $S^{(n)} \oplus S^* \notin \mathbb{A}_{n+2}$ induces easily a contraction for the second part. □

3. Matrices of dual algebras with properties $(\mathbb{A}_{m,n})$

In this section we discuss dual algebras $\mathcal{M}_n(\mathcal{A})$ and properties $(\mathbb{A}_{m,n})$. The following theorem is an improvement of [1, Proposition 1.3] (or [11, Lemma 3.3]).

THEOREM 3.1 *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra. Suppose that $k \in \mathbb{N}$ and $1 \leq m, n \leq \aleph_0$. Then the dual algebra $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbb{A}_{m,n})$ if and only if $\mathcal{A}$ has property $(\mathbb{A}_{km,kn})$.*

Proof. We shall prove this theorem in the case of $1 \le m, n < \aleph_0$ because those of other cases are similar with that. Assume that the dual algebra $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbf{A}_{m,n})$. If we give a set $\{[L_{ij}]\}_{\substack{0 \le i < km \\ 0 \le j < kn}} \subset \mathcal{Q}_{\mathcal{A}}$, then there exist $\widetilde{x}_i, \widetilde{y}_j \in \mathcal{H}^{(k)}$, $1 \le i \le m$ $1 \le j \le n$ such that

$$[\widetilde{L_{ij}}] = [\widetilde{x}_i \otimes \widetilde{y}_j], \tag{3.1}$$

where $[\widetilde{L_{ij}}]$ denotes the transpose of the following matrix:

$$\begin{pmatrix} [L_{(i-1)k+1,(j-1)k+1}] & [L_{(i-1)k+1,(j-1)k+2}] & \cdots & [L_{(i-1)k+1,jk}] \\ [L_{(i-1)k+2,(j-1)k+1}] & [L_{(i-1)k+2,(j-1)k+2}] & \cdots & [L_{(i-1)k+2,jk}] \\ \vdots & \vdots & \ddots & \vdots \\ [L_{ik,(j-1)k+1}] & [L_{ik,(j-1)k+2}] & \cdots & [L_{ik,jk}]. \end{pmatrix}$$

Now if we say $\widetilde{x}_i = x_{1i} \oplus \cdots \oplus x_{ki}$, $1 \le i \le m$ and $\widetilde{y}_j = y_{1j} \oplus \cdots \oplus y_{kj}$, $1 \le j \le n$, then we have

$$\begin{aligned} [\widetilde{x}_i \otimes \widetilde{y}_j] &= [(x_{1i} \oplus \cdots \oplus x_{ki}) \otimes (y_{1j} \oplus \cdots \oplus y_{kj})] \\ &= \begin{pmatrix} [x_{1i} \otimes y_{1j}] & \cdots & [x_{ki} \otimes y_{1j}] \\ \vdots & \ddots & \vdots \\ [x_{1i} \otimes y_{kj}] & \cdots & [x_{ki} \otimes y_{kj}] \end{pmatrix} \end{aligned} \tag{3.2}$$

For a convenient notation we write

$$\begin{cases} u_1 = x_{11}, & \cdots, \quad u_k = x_{k1} \\ u_{k+1} = x_{12}, & \cdots, \quad u_{2k} = x_{k2} \\ \vdots & \\ u_{(m-1)k} = x_{1m}, & \cdots, \quad u_{mk} = x_{km}. \end{cases}$$

and

$$\begin{cases} v_1 = y_{11}, & \cdots, \quad v_k = y_{1k} \\ v_{k+1} = y_{21}, & \cdots, \quad v_{2k} = y_{2k} \\ \vdots & \\ v_{(n-1)k} = y_{n1}, & \cdots, \quad v_{nk} = y_{nk} \end{cases}$$

Then it is easy to show that $[L_{ij}] = [u_i \otimes v_j]$ for $1 \le i \le km$, $1 \le j \le kn$, which implies that $\mathcal{A}$ has property $(\mathbb{A}_{km,kn})$.

On the other hand, the calculation for the converse implication is similar with that of the above. So we will omit it here. Hence the proof is complete. □

By applying Theorem 3.1 and elementary facts of properties $(\mathbb{A}_{m,n})$, we obtain the following theorem without difficulties.

THEOREM 3.2. *Suppose $k, n \in \mathbb{N}$ and a dual algebra $\mathcal{A}$ has property $(\mathbb{A}_{kn,\aleph_0})$ but not property $(\mathbb{A}_{kn+1})$. Then we have*
(1) $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbb{A}_{n,\aleph_0})$ but not property $(\mathbb{A}_{n+1})$
and
(2) $\mathcal{M}_{k+1}(\mathcal{A})$ doesn't have property $(\mathbb{A}_n)$.

The following is an immediate corollary of Theorem 3.2.

COROLLARY 3.3. *Let S be the unilateral shift operator of multiplicity one. Suppose $m, n \in \mathbb{N}$. Then the dual algebra $\mathcal{M}_n(\mathcal{A}_{S^{(mn)}})$ has property $(\mathbb{A}_{m,\aleph_0})$ but not property $(\mathbb{A}_{n+1})$.*

Finally, we close this paper with an open problem. The following conjecture comes from Professor Carl Pearcy.

CONJECTURE 3.4. $S \oplus S^* \notin \mathbb{A}_2$.

This implies that $\mathbb{A}_{1,2} \cap \mathbb{A}_{2,1} \neq \mathbb{A}_2$. Moreover, Conjecture 3.4 can be restated with that $\mathcal{M}_2(\mathcal{A}_{S \oplus S^*})$ does not have property $(\mathbb{A}_1)$. According to Theorem 3.1 we have that $\mathcal{M}_n(\mathcal{A}_{S^{(n)} \oplus S^*})$ has property $(\mathbb{A}_1)$ but not property $(\mathbb{A}_2)$ for $n \ge 2$. Since $S^{(n)} \oplus S^* \notin \mathbb{A}_{n+2}$, it is obvious that $\mathcal{M}_{n+2}(\mathcal{A}_{S^{(n)} \oplus S^*})$ does not have property $(\mathbb{A}_1)$. But we don't know whether the dual algebra $\mathcal{M}_{n+1}(\mathcal{A}_{S^{(n)} \oplus S^*})$ has property $(\mathbb{A}_1)$ for $n \ge 1$.

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