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LINEAR OPERATORS THAT PRESERVE SPANNING COLUMN RANKS OF NONNEGATIVE MATRICES*

SUK-GEUN HWANG, SI-JU KIM AND SEOK-ZUN SONG

1. Introduction

If $\mathbf{S}$ is a semiring of nonnegative reals, which linear operators T on the space of $m \times n$ matrices over $\mathbf{S}$ preserve the column rank of each matrix? Evidently if P and Q are invertible matrices whose inverses have entries in $\mathbf{S}$, then $T : X \to \mathrm{PXQ}$ is a column rank preserving, linear operator. Beasley and Song obtained some characterizations of column rank preserving linear operators on the space of $m \times n$ matrices over $\mathbf{Z}_+$, the semiring of nonnegative integers in [1] and over the binary Boolean algebra in [7] and [8]. In [4], Beasley, Gregory and Pullman obtained characterizations of semiring rank-1 matrices and semiring rank preserving operators over certain semirings of the nonnegative reals. We consider some results in [4] in view of a certain column rank instead of semiring rank.

In this paper, we define "spanning column rank" (see Section 2), which is the same as column rank on the space of matrices over a field or $\mathbf{Z}_+$ but differs from column rank in general semirings. We obtain a characterization of spanning column rank 1 matrices. We also characterize linear operators which preserve the spanning column ranks of matrices over the nonnegative part of a unique factorization domain that is not a field in the reals.

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2. Definitions and basic properties of spanning column rank

Let $\mathbf{S}$ be any subset of $\mathbf{R}_+$(the nonnegative reals). We'll call it a ***nonnegative semidomain*** if it contains 0,1 and is closed under multiplication and addition (the usual operations). If $\mathbf{D}$ is a subring of $\mathbf{R}$ containing 1 (so $\mathbf{D}$ is an integral domain), let $\mathbf{D}_+$ denote the set of its nonnegative elements. Then $\mathbf{D}_+$ is a nonnegative semidomain. Examples are $\mathbf{R}_+, \mathbf{Q}_+, \mathbf{Z}_+, (\mathbf{Z}[\sqrt{2}])_+$ etc., where $\mathbf{Q}$ denotes the rationals and $\mathbf{Z}$ the integers. Note that $(\mathbf{Z}[\sqrt{2}])_+$ contains $Z_+[\sqrt{2}]$ properly, since e.g. $\sqrt{2}-1$ is in the left member but not the right. There are other nonnegative semidomains : e.g. $\mathbf{H} = \{0,1,2,3\} \cup \{q \in \mathbf{Q} \mid q \geq 4\}$ is not of the form $\mathbf{D}_+$ for any integral domain $\mathbf{D}$ in $\mathbf{R}$.

Hereafter, unless otherwise specified, $\mathbf{S}$ will denote an arbitrary nonnegative semidomain.

Let $\mathbf{A}$ be an $m \times n$ matrix over $\mathbf{S}$. If A is a nonzero matrix, then the ***semiring rank*** [1] of $A, r_{\mathbf{S}}(A)$, is the least k for which there exist $m \times k$ and $k \times n$ matrices F and G over $\mathbf{S}$ such that A=FG. The zero matrix is assigned the semiring rank 0. The set of $m \times n$ matrices with entries in $\mathbf{S}$ is denoted by $\mathbb{M}_{m,n}(\mathbf{S})$. Addition, multiplication by scalars, and the product of matrices are defined as if $\mathbf{S}$ were a field.

If $\mathbb{V}$ is a nonempty subset of $S^k \equiv \mathbb{M}_{k,1}(\mathbf{S})$ that is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbf{S}$. The notions of subspace and of spanning sets are the same as if $\mathbf{S}$ were a field.

A set $\mathbb{A}$ of vectors over $\mathbf{S}$ is *linearly dependent* if for some $\mathbf{a} \in \mathbb{A}$, $\mathbf{a}$ is a linear combination of the vectors in $\mathbb{A} - \{\mathbf{a}\}$. Otherwise $\mathbb{A}$ is *linearly independent.*

As with fields, a *basis* for a vector space $\mathbb{V}$ is a spanning subset of least cardinality. That cardinality is the *dimension,* $\dim(\mathbb{V})$, of $\mathbb{V}$.

The *column space* of an $m \times n$ matrix A over $\mathbf{S}$ is the vector space that is spanned by its columns. The *column rank*, $c_{\mathbf{S}}(A)$, of a nonzero $m \times n$ matrix A over $\mathbf{S}$ is the dimension of its column space. The ***spanning column rank***, $sc_{\mathbf{S}}(A)$, is the minimum number of the columns of A which span its column sace. As with semiring rank, the zero matrix is assigned column rank and spanning column rank 0.

It follows that

$$0 \leq r_{\mathbf{S}}(A) \leq c_{\mathbf{S}}(A) \leq sc_{\mathbf{S}}(A) \leq n \tag{2.1}$$

for all $m \times n$ matrices A over $\mathbf{S}$.

Over a field $\mathbf{F}$ we have $c_{\mathbf{F}}(A) = sc_{\mathbf{F}}(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbf{F})$. For, if $c_{\mathbf{F}}(A) = k$, then the column space of A has dimension k. So any r columns of A are linearly dependent for r which is greater than k. Hence $sc_F(A) \leq$k. Therefore the column rank and spanning column rank functions are equal over any field by (2.1).

We can also show that the column rank and spanning column rank are the same over $\mathbb{M}_{m,n}(Z_+)$ or $\mathbb{M}_{m,n}(\mathbf{B})$, where $\mathbf{B}$ is the two element Boolean algebra. But they may differ over other semirings. The spanning column rank may actually exceed its column rank over some semirings.

For example, consider $A = [3 - \sqrt{7}, \sqrt{7} - 2]\, over\, \mathbf{S} = (\mathbf{Z}[\sqrt{7}])_+$. Since $(3 - \sqrt{7}) + (\sqrt{7} - 2) = 1, \{1\}$ is a spanning set of the column space of A. So $c_{\mathbf{S}}(A) = 1$. But $sc_{\mathbf{S}}(A) = 2$ since $3 - \sqrt{7} \neq a(\sqrt{7} - 2)$ and $\sqrt{7} - 2 \neq a(3 - \sqrt{7})$ for any a in $\mathbf{S}$.

Here are some basic properties of spanning column rank.

If the columns of $A \in \mathbb{M}_{m,n}(\mathbf{S})$ are linearly independent over $\mathbf{S}$, then

$$sc_{\mathbf{S}}(A) = n. \tag{2.2}$$

If $\mathbf{S} \subseteq T$, then $sc_{\mathbf{S}}(A) \geq sc_{\mathbf{T}}(A)$ for arbitrary A with entries in $\mathbf{S}$. (2.3)

Let A and B be matrices over $\mathbf{S}$. Then

$$sc_{\mathbf{S}}(AB) \leq sc_{\mathbf{S}}(B). \tag{2.4}$$

For, assume that $sc_{\mathbf{S}}(B) = $ k. Then without loss of generality, we may assume that $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a set of columns of B of minimum cardinality that spans its column space. Since the jth column of AB is of the form $A\mathbf{b}_j$ and $\mathbf{b}_j = \sum_{i=1}^{k} s_i \mathbf{b}_i$ for some $s_i \in \mathbf{S}$, we have $A\mathbf{b}_j = \sum_{i=1}^{k} s_i A\mathbf{b}_i$. That is, any column of AB can be written as a linear combination of $A\mathbf{b}_1, \ldots, A\mathbf{b}_k$. Hence $sc_{\mathbf{S}}(AB) \leq k = sc_{\mathbf{S}}(B)$.

But, in general $sc_{\mathbf{S}}(AB)$ is not less than $sc_{\mathbf{S}}(A)$ as shown in the following example : let

$$A = [3, 7, 7],\ B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

be matrices over $\mathbf{S} = \mathbf{Z}_+$. Then $sc_{\mathbf{S}}(AB) = sc_{\mathbf{S}}([3, 10, 17]) = 3$, but $sc_{\mathbf{S}}(A) = 2$.

If B is obtained by deleting some rows of A, then

$$sc_{\mathbf{S}}(B) \leq sc_{\mathbf{S}}(A). \tag{2.5}$$

But suppressing a column may increase the spanning column rank. For example, consider

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

over $\mathbf{B}$, the two element Boolean algebra. Then $sc_{\mathbf{B}}(A) = 3$ because the last three columns are linearly independent and span the column space of A. We delete column 5 from A to obtain A', then $sc_{\mathbf{B}}(A') = 4$ since the four columns of A' are linearly independent.

If V, V^{-1} have all entries in $\mathbf{S}$, then

$$sc_{\mathbf{S}}(VA) = sc_{\mathbf{S}}(A) \text{ by } (2.4). \tag{2.6}$$

If X is a matrix over $\mathbf{S}$ and $X = \mathbf{a}\mathbf{x}^t$, then $\mathbf{a}, \mathbf{x}$ are called *left* and *right factors* of X respectively. Both $\mathbf{a}$ and $\mathbf{x}$ are referred to as *factors* of X. In particular, $\mathbf{a}$ is called a *strong right factor* of X if $\mathbf{a}^t$ has spanning column rank 1. For $X \in \mathbf{M}_{m,n}(\mathbf{S})$, we write $sc(X)$ for $sc_{\mathbf{S}}(X), c(X)$ for $c_{\mathbf{S}}(X)$ and $r(X)$ for $r_{\mathbf{S}}(X)$.

LEMMA 2.1. *For $A \in \mathbf{M}_{m,n}(\mathbf{S}), sc(A) = 1$ if and only if A can be factored as $\mathbf{x}\mathbf{a}^t$ for some $\mathbf{a} \in \mathbf{S}^n, \mathbf{x} \in \mathbf{S}^m$, where $\mathbf{x} \neq 0$ and $\mathbf{a}$ is a strong right factor.*

Proof. If $sc(A) = 1$, then there exists one column $\mathbf{a}_k$ of A such that all the other columns $\mathbf{a}_i$ are expressed as a scalar multiple of $\mathbf{a}_k$, that is $\mathbf{a}_i = \alpha_i \mathbf{a}_k$ for some $\alpha_i \in \mathbf{S}$. Therefore $A = \mathbf{a}_k[\alpha_1, \dots, \alpha_n]$ with $\alpha_k = 1$. Let $\mathbf{x} = \mathbf{a}_k, \mathbf{a}^t = [\alpha_1, \dots, \alpha_n]$. Then we have done. The converse is clear. □

In the sequel, let $\mathbf{U}_+$ be the nonnegative part of a unique factorization domain $\mathbf{U}$ which is not a field in $\mathbf{R}$ - for example $\mathbf{R}_+ \cap \mathbf{Q}[e]$ with transcendental e, etc.

LEMMA 2.2. *There are an infinite number of primes in* $\mathbf{U}_+$.

Proof. If there is no primes in $\mathbf{U}_+$, then $\mathbf{U}$ is a field, which is not the case. So there is at least one prime in $\mathbf{U}_+$. Let $p_1, p_2, \ldots,$ be primes in $\mathbf{U}_+$, and suppose that there is a last prime ; call it p_n. Now consider the positive element

$$Q = p_1 p_2 \cdots p_n + 1.$$

Since $Q \in \mathbf{U}_+$, there are a unit α and primes $q_1, \ldots, q_s$ in $\mathbf{U}_+$ such that Q has the form $\alpha q_1 \cdots q_s$. But $p_1, p_2, \ldots, p_n$ are the only primes, so that q_1 must be equal to one of $p_1, p_2, \ldots, p_n$. Since q_1 divides both $p_1 p_2 \cdots p_n$ and Q, we arrive at q_1 divides $Q - p_1 p_2 \cdots p_n$ or, equivalently, q_1 divides 1. Then q_1 is a unit, which is a contradiction. Thus the number of primes is infinite. □

3. Spanning column rank preserving linear operators

If T maps $\mathbb{M}_{m,n}(\mathbf{S})$ into itself and $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $\alpha, \beta \in \mathbf{S}$ and for all $X, Y \in \mathbb{M}_{m,n}(\mathbf{S})$, then T is a *linear operator* on $\mathbb{M}_{m,n}(\mathbf{S})$.

In [4], it was shown that if $S = \mathbf{U}_+$ and T preserves real or semiring ranks of matrices over $\mathbf{U}_+$, then there exist nonsingular matrices U and V such that $T(X)$=UXV (or possibly $T(X) = UX^tV$ if $m = n$) for all $X \in \mathbb{M}_{m,n}(\mathbf{U}_+)$.

In [1], we obtained a characterization of linear operators which preserve column ranks on $\mathbb{M}_{m,n}(\mathbf{Z}_+)$. Since the column rank and spanning column rank are the same on $\mathbb{M}_{m,n}(\mathbf{Z}_+)$, the characterization holds for spanning column ranks.

In this section we shall obtain a similar characterization of linear operators which preserve spanning column ranks on $\mathbb{M}_{m,n}(\mathbf{U}_+)$. We say that a linear operator T *preserves spanning column rank* k provided that $sc(T(X)) = k$ if $sc(X) = k$, where $X \in \mathbb{M}_{m,n}(\mathbf{S})$.

Let $\mathbf{e}_i$ be the vector in $\mathbf{S}^n$ with a "1" in the ith position and zero elsewhere. We say that X is a *column matrix* if $X = \mathbf{x}\mathbf{e}_i^t$ for some $1 \leq i \leq n$ and some vector $\mathbf{x} \in \mathbf{S}^m$.

LEMMA 3.1. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbf{U}_+), n \geq 2$. If T preserves spanning column ranks 1 and 2, then T maps column matrices to column matrices.*

Proof. Suppose to the contrary that T maps a column matrix to a matrix which is not a column matrix. Say $\mathbf{x}(\mathbf{e}_1)^t = X_1$ and $T(X_1)$ has more than one nonzero column. For each $1 \leq i \leq n$, let $X_i = \mathbf{x}(\mathbf{e}_i)^t$. Let $S = \{1, \ldots, n\}$ and let $S_1 = \{j$:the jth column of $T(X_i)$ is zero for all $1 \leq i \leq n\}$. Then for each $i \in S - S_1$, there is a j_1 such that the ith column of $T(X_{j(i)})$ is not zero. Now $T(X_1)$ has at least two nonzero columns, say columns k_1 and k_2. Let $S_2 = S - S_1 - \{k_1, k_2\}$, and let $X = X_1 + \sum_{i \in S_2} X_{j(i)}$. Note that for any $k \in S - S_1$ the kth column of $T(X)$ is nonzero. Further, since X consists of at most $n-1$ distinct summands, each of which is a column matrix, there is at least one zero column in X, say the ith. Let $Y = X_i$. Since $T(X)$ has zero columns only corresponding to indices in S_1(where $T(Y)$ also must have a zero column) we can restrict our attention to those columns in $T(X)$ that are nonzero; hence we lose no generality in assuming that $T(X)$ has no zero column. Thus, since X, and hence $T(X)$, has spanning column rank $1, T(X) = \mathbf{u}\mathbf{a}^t$, where $\mathbf{a}^t = [a_1, \ldots, a_n]$ has all nonzero entries. Let $T(Y) = \mathbf{v}\mathbf{b}^t$ with $\mathbf{b}^t = [b_1, \ldots, b_n]$. Since $\mu X + Y$ has spanning column rank 1 for arbitrary μ in $\mathbf{U}_+$,

$$T(\mu X + Y) = [\mu a_1 \mathbf{u} + b_1 \mathbf{v} \mid \mu a_2 \mathbf{u} + b_2 \mathbf{v} \mid \ldots \mid \mu a_n \mathbf{u} + b_n \mathbf{v}]$$

has also spanning column rank 1. Now we consider two cases:
Case 1) If $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in $\mathbf{U}_+$, then we have, for some fixed j,

$$\mu a_k \mathbf{u} + b_k \mathbf{v} = r_k(\mu a_j \mathbf{u} + b_j \mathbf{v})$$

for some $r_k \in \mathbf{U}_+, k = 1, \ldots,$ n. If $a_k \neq r_k a_j$ for some k, then

$$\mu \mid a_k - r_k a_j \mid \mathbf{u} = \mid r_k b_j - b_k \mid \mathbf{v},$$

which is a contradiction to the condition that $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in $\mathbf{U}_+$. Thus $a_k = r_k a_j$ and $b_k = r_k b_j, k = 1, \ldots,$ n. That is, $\mathbf{a} = a_j \mathbf{r}$ and $\mathbf{b} = b_j \mathbf{r}$, where $\mathbf{r}^t = [r_1, \ldots, r_n]$ with $r_j = 1$.
Case 2) Assume that $c\mathbf{u} = d\mathbf{v}$ for some c, d in $\mathbf{U}_+$. By Lemma 2.2, we can choose a prime μ such that μ does not divide all nonzero $b_i, i = 1, \ldots,$ n. Consider

$$\begin{aligned} T(c\mu^h X + Y) &= [\mu^h a_1 c\mathbf{u} + b_1 \mathbf{v} | \mu^h a_2 c\mathbf{u} + b_2 \mathbf{v} | \cdots \mu^h a_n c\mathbf{u} + b_n \mathbf{v}] \\ &= \mathbf{v}[\mu^h a_1 d + b_1, \mu^h a_2 d + b_2, \ldots, \mu^h a_n d + b_n], \end{aligned}$$

which has spanning column rank 1 for any positive integer h. Since the columns of $T(c\mu^h X + Y)$ are finite in number, there exists a column j and a sequence of h's with the properties that i) the jth columns of $T(c\mu^h X + Y)$ spans the column space for each h, and ii) the difference between two successive terms in the sequence is at most n. Therefore for infinitely many h

$$\mu^h a_k d + b_k = r_{hk}(\mu^h a_j d + b_j) \tag{1}$$

for some $r_{hk} \in \mathbf{U}_+, k = 1, \ldots,$ n. In (1), if $b_j = 0$ then b_k must be divided by infinitely many nonunit μ^h. But it is impossible since μ does not divide b_k for at least one nonzero b_k. So b_j is not zero. If the column space of $T(c\mu^g X + Y)$ is spanned by its jth column, then we get

$$\mu^g a_k d + b_k = r_{gk}(\mu^g a_j d + b_j) \tag{2}$$

for some $r_{gk} \in \mathbf{U}_+$. From (1) and (2), we get $| r_{gk} - r_{hk} | \in \mathbf{U}_+$ for $g > h$. By the choice of μ, we can choose infinitely many pairs h and g such that they satisfy $h < g \le h + n$ and the column spaces of $T(c\mu^h X + Y)$ and $T(c\mu^g X + Y)$ are spanned by their jth column respectively. For such pairs h and g, consider

$$\begin{aligned} |r_{gk} - r_{hk}| &= \left| \frac{(\mu^g a_k d + b_k)}{(\mu^g a_j d + b_j)} - \frac{(\mu^h a_k d + b_k)}{(\mu^h a_j d + b_j)} \right| \\ &= \frac{d \mid (a_k b_j - a_j b_j)(\mu^{g-h} - 1) \mid \mu^h}{(\mu^g a_j d + b_j)(\mu^h a_j d + b_j)} \end{aligned} \tag{3}$$

Assume that $r_{gk} \neq r_{hk}$ for all such pairs h and g. Since μ is prime, μ is not divided by $\mu^h a_j d + b_j$. If $\mu^h a_j d + b_j$ has μ as its factor, then $\mu^h a_j d + b_j = \beta\mu$ for some $\beta \in \mathbf{U}_+$. Thus $\mu(\beta - \mu^{h-1} a_j d) = b_j$ and hence b_j is divided by μ, which is a contradiction. Then μ^h does not have any factor of $(\mu^h a_j d + b_j)(\mu^g a_j d + b_j)$. Since $d \mid a_k b_j - a_j b_k \mid$ is fixed and $\mid \mu^{g-h} - 1 \mid$ is finite value for any pairs h and g with $1 \le g - h \le n$, the prime factors of $d \mid (a_k b_j - a_j b_k)(\mu^{g-h} - 1) \mid$ are finite in number. Thus we can choose sufficiently large pair h and g with $1 \le g - h \le n$ such that $d \mid (a_k b_j - a_j b_k)(\mu^{h-g} - 1) \mid$ does not contain some prime factors of $(\mu^h a_j d + b_j)(\mu^g a_j d + b_j)$. Then the denominator of (3) contains some nonunit prime factors such that the numerator of (3) does not contain. Since $\mathbf{U}_+$ contains no element of the form x/y, where y has a prime factor which x does not, the fractional expression of (3) is not an element of $\mathbf{U}_+$. Thus we have a contradiction such that $| r_{gk} - r_{hk} | \notin \mathbf{U}_+$ for some pair g and h with $h < g \leq h$+n. Hence $r_{gk} = r_{hk}$ for some h and g. Subtracting (1) from (2), we get $a_k = r_{hk} a_j$ for all $k = 1, \ldots,$ n. And we get $b_k = r_{hk} b_j$ for all $k = 1, \ldots, n$ from (1). That is, $\mathbf{a} = a_j \mathbf{r}$ and $\mathbf{b} = b_j \mathbf{r}$ where $\mathbf{r} = [r_{h1}, \ldots, r_{hn}]$ with $r_{hj} = 1$.

By cases 1) and 2), $T(X)$ and $T(Y)$ has the same strong right factor $\mathbf{r}$. Thus $sc(T(\alpha X + \beta Y)) = sc(\alpha a_j \mathbf{u} + \beta b_j \mathbf{v})\mathbf{r}^t = 1$ for arbitrary α, β in $\mathbf{U}_+$. This contradicts that T preserves spanning column rank 2 since $\alpha X + \beta Y$ has spanning column rank 2 for relatively prime α and β in $\mathbf{U}_+$. Hence T maps column matrices to column matrices. □

We use the notation E_{ij} for the $m \times n$ matrix whose (i,j)-entry is 1 and whose other entries are all 0. We also write $\mathbf{x}_j$ for the jth column of X.

PROPOSITION 3.2. *Let A be an $m \times n$ matrix over* $\mathbf{S}$ *and* $T(X) = AX$ *for all* X *in* $\mathbb{M}_{n,k}(\mathbf{S})$.

(1) If T preserves spanning column rank 1, then $T(X) = 0$ *only if* $X = 0$.

(2) If $k \geq 2$, and T preserves spanning column ranks 1 and 2, then T is injective on $\mathbb{M}_{n,k}(\mathbf{S})$.

Proof. (1) If $AX = \mathbf{0}$ and $X \neq \mathbf{0}$, then there exists $x_{ij} \neq 0$ for some i, j. Then $0 = (AX)_{kj} = \sum_{t=1}^{n} a_{kt} x_{tj}$ for arbitrary k. Hence $a_{ki} = 0$ for all k. That is, $\mathbf{a}_i$(the ith column of A) is zero. We may take $X = E_{ij}$. Then $sc(X) = 1$ but $T(X)$ is a zero matrix and has spanning column rank 0. This contradicts the fact that T preserves spanning column rank 1.

(2) Suppose $T(B) = T(C)$. Then for all j, $A\mathbf{b}_j = A\mathbf{c}_j \equiv \mathbf{z}_j$. If $\mathbf{z}_j = \mathbf{o}$ then it follows from (1) that $\mathbf{b}_j = \mathbf{o} = \mathbf{c}_j$. If $\mathbf{z}_j \neq \mathbf{o}$, let $Y = [\mathbf{b}_j \mid \mathbf{c}_j \mid \mathbf{o} \mid \ldots \mid \mathbf{o}]$. Then $sc(T(Y)) = sc(AY) = sc([\mathbf{z}_j \mid \mathbf{z}_j \mid \mathbf{o} \mid \ldots \mid \mathbf{o}]) = 1$, but T preserves spanning column rank 2, so $sc(Y) = 1$. Therefore $\mathbf{b}_j = \alpha \mathbf{c}_j$ or $\mathbf{c}_j = \beta \mathbf{b}_j$ for some α, β. If $\mathbf{b}_j = \alpha \mathbf{c}_j$, then $A\mathbf{c}_j = \mathbf{z}_j = A\mathbf{b}_j = \alpha A\mathbf{c}_j$ and $\mathbf{z}_j \neq \mathbf{o}$. So $\alpha = 1$ and $\mathbf{b}_j = \mathbf{c}_j$ for all j. For $\mathbf{c}_j = \beta \mathbf{b}_j$, we have the same results. Thus T is injective on $\mathbb{M}_{n,k}(\mathbf{S})$.□

EXAMPLE 3.3. Let $T_k(X) = (\sum_{i,j} x_{ij})A$ for all $X \in \mathbb{M}_{m,n}(\mathbf{S})$, where $A \in \mathbb{M}_{m,n}(\mathbf{S})$ and $sc(A)$=k. Then T_k preserves spanning column rank k, but T_k is not injective. Thus the condition (2) in Proposition 3.2 can not be relaxed by requiring that T preserves spanning column rank 1 or T preserves spanning column rank 2. □

EXAMPLE 3.4. Let $T(E_{12}) = E_{22}, T(E_{22}) = E_{12}$ and $T(E_{ij}) = E_{ij}$ for all other i, j. Extend T to $\mathbb{M}_{m,n}(\mathbf{S})$ by linearity. Let $A = E_{11} + E_{12}$ and $B = E_{11} + E_{22}$. Then $sc(A) = 1$ but $sc(T(A)) = 2$. Also $sc(B) = 2$ but $sc(T(B)) = 1$. Nevertheless, T is injective. In fact, T is bijective. Thus injectivity alone does not ensure that spanning column ranks 1, 2 will be preserved by a linear operator.□

THEOREM 3.5. *For $n \geq 2$, let T be a linear operator on $\mathbb{M}_{m,n}(\mathbf{U}_+)$. Then T preserves spanning column ranks 1 and 2 if and only if there exist $Q \in \mathbb{M}_{m,m}(\mathbf{U}_+), D$ and $P \in \mathbb{M}_{n,n}(\mathbf{U}_+)$ such that $T(X)$=$QXDP$ for all $X \in \mathbb{M}_{m,n}(\mathbf{U}_+)$, where Q is invertible in $\mathbb{M}_{m,m}(\mathbf{R})$, D is an invertible diagonal matrix whose diagonal entries are all units in $\mathbf{U}_+$ and P is a permutation matrix.*

Proof. We first show the sufficiency. Suppose $T(X)$= QXDP and X has spanning column rank 1, say $X = \sigma\mathbf{x}\mathbf{a}^t$ for some nonzero σ in $\mathbf{U}_+$ and $a_i = 1$ for some i. Let $\mathbf{P}^t$ correspond to $\pi \in S_n$ and $i = \pi(j)$. Then QXDP= $\sigma Q\mathbf{x}(P^tD\mathbf{a})^t$ and $d_{\pi(j)}a_{\pi(j)} = d_{\pi(j)}$, which is a diagonal entry of D and is a unit element in $\mathbf{U}_+$. Hence QXDP has spanning column rank 1. Then T preserves spanning column rank 1. If X has spanning column rank 2, then X has two linearly independent columns which span all other columns of X. Without loss of generality, we may assume that they are $\mathbf{x}_1$ and $\mathbf{x}_2$. Then $Q\mathbf{x}_1$ and $Q\mathbf{x}_2$ span all the other columns of QX. Now $sc(T(X)) = 1$ or 2 by the facts that multiplying DP on the right hand of QX does not change column rank of QX and that T preserves column rank 1. If $sc(T(X)) = 1$, then $sc([Q\mathbf{x}_1, Q\mathbf{x}_2]) = 1$. Hence either $Q\mathbf{x}_1 = r\,Q\mathbf{x}_2$ or $Q\mathbf{x}_2 = r\,Q\mathbf{x}_1$ for some $r \in \mathbf{U}_+$. Thus $\mathbf{x}_1 = r\mathbf{x}_2$ or $\mathbf{x}_2 = r\mathbf{x}_1$ in $\mathbf{R}$ and hence in $\mathbf{U}_+$ since Q is invertible in $\mathbf{R}$. Then $sc(X) = 1$, which is a contradiction. Hence $sc(T(X)) = 2$ and T preserves spanning column rank 2.

Conversely, suppose T preserves spanning column ranks 1 and 2. Let $X_i = \mathbf{x}(\mathbf{e}_i)^t, i = 1, \ldots, n$, for some fixed $\mathbf{x} \in (\mathbf{U}_+)^m$. By Lemma 3.1, $T(X_i) = \mathbf{y}_i(\mathbf{e}_{\pi(i)})^t$ where $(\mathbf{y}_i)^t = [y_{1i}, y_{2i} \ldots, y_{mi}]$ is dependent on $\mathbf{x}$ and $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. If π is not one-to-one, then for some i and $j, \alpha T(X_i) + \beta T(X_j)$ has only one nonzero column for all α, β. That is, $T(\alpha X_i + \beta X_j)$ has spanning column rank 1 for all α, β, a contradiction since $\alpha X_i + \beta X_j$ has spanning column rank 2 for relatively prime α, β in $\mathbf{U}_+$. Thus π is one-to-one and hence it is a permutation. So without loss of generality, we assume $\pi = \epsilon$, the identity permutation, so that $T(X_1) = \mathbf{y}_1(\mathbf{e}_1)^t$ and $T(X_2) = \mathbf{y}_2(\mathbf{e}_2)^t$. If $y_{h1} \neq 0$ and $y_{h2} = 0$ or vice versa, then $T(X_1 + X_2) = \mathbf{y}_1(\mathbf{e}_1)^t + \mathbf{y}_2(\mathbf{e}_2)^t$ has spanning column rank 2, contradicting that T preserves spanning column ranks 1 and 2 since $X_1 + X_2$ has spanning column rank 1. Thus $y_{h1} = 0$ if and only if $y_{h2} = 0$. We assume without loss of generality that y_{11} and y_{12} are positive. Since $X_1 + X_2$ has spanning column rank $1, \mathbf{y}_2 = d_2\mathbf{y}_1$ or $\mathbf{y}_1 = d_2\mathbf{y}_2$ for some d_2 in $\mathbf{U}_+$. Let $\mathbf{y}_2 = d_2\mathbf{y}_1$. If d_2 is not a unit, choose p relatively prime to d_2 in $\mathbf{U}_+$, then

$$\begin{aligned} T(pX_1 + X_2) &= [p\mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{o} \mid \ldots \mid \mathbf{o}] \\ &= [p\mathbf{y}_1 \mid d_2\mathbf{y}_1 \mid \mathbf{o} \mid \ldots \mid \mathbf{o}] \end{aligned}$$

has spanning column rank 2 while $pX_1 + X_2$ has spanning column rank $1, a$ contradiction. Thus d_2 is a unit. For the case $\mathbf{y}_1 = d_2\mathbf{y}_2$, we can show that d_2 is a unit by a similar argument. Then $\mathbf{y}_2 = (d_2)^{-1}\mathbf{y}_1$. It follows that $T(X_i) = (d_i\mathbf{y}_1)(\mathbf{e}_i)^t$ for some unit $d_i, i = 1, \ldots,$ n. In particular, when $X_i = E_{ji}$, there exists some vector $\mathbf{y}_j^1 \equiv \mathbf{u}_j = [y_{1j}, y_{2j}, \cdots, y_{mj}]^t \in \mathbb{M}_{m,1}(\mathbf{U}_+)$ and some unit $d_i \in \mathbf{U}_+$ such that

$$T(E_{ji}) = (d_i\mathbf{u}_j)(\mathbf{e}_i)^t = \mathbf{u}_j(d_i\mathbf{e}_i)^t$$

for all i, j. Let Q be the matrix $[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_m]$ and D be the diagonal matrix diag$(d_1, d_2, \ldots, d_n)$. Then for an arbitrary $X \in \mathbb{M}_{m,n}(\mathbf{U}_+)$,

$$T(X) = \sum_{j=1}^{m}\sum_{i=1}^{n} x_{ji}T(E_{ji}) = \sum_{j=1}^{m}\sum_{i=1}^{n} x_{ji}\mathbf{u}_j(d_i\mathbf{e}_i)^t.$$

So the (s,t) entry of $T(X)$ is $\sum_{j=1}^{m} x_{jt}y_{sj}d_t$. The (s,t) entry of QXD is $\sum_{j=1}^{m} y_{sj}x_{jt}d_t$, which is the (s,t) entry of $T(X)$. Thus $T(X)$=QXD for all $X \in \mathbb{M}_{m,n}(\mathbf{U}_+)$.

Further we show that Q is nonsingular in $\mathbb{M}_{m,m}(\mathbf{R})$. Suppose that $Q = (q_{ij})$ is singular. Say, $Q\mathbf{x} = \mathbf{o}$ for some nonzero $\mathbf{x}$ in $\mathbb{M}_{m,1}(\mathbf{R})$. Since $\mathbf{x}$ can be considered as a solution of the homogeneous system of linear equations with coefficients $q_{ij} \in \mathbf{U}_+$, we may assume, without loss of generality, that the entries of $\mathbf{x}$ are all in $\mathbf{U}$. So let $\alpha = 1 + \max_{1 \le i \le m} | x_i |$, and $\mathbf{z} = \alpha\mathbf{j} + \mathbf{x}$, where $\mathbf{j}$ is the vector of all 1's. Then $\mathbf{z} \in \mathbb{M}_{m,1}(\mathbf{U}_+)$ and $Q\mathbf{z} = Q(\alpha\mathbf{j} + \mathbf{x}) = Q(\alpha\mathbf{j})$. Thus

$$T(\mathbf{z}\mathbf{e}_1^t + \alpha\mathbf{j}\mathbf{e}_2^t) = Q(\mathbf{z}\mathbf{e}_1^t)D + Q(\alpha\mathbf{j}\mathbf{e}_2^t)D$$
$$= Q(\alpha\mathbf{j})\mathbf{e}_1^t D + Q(\alpha\mathbf{j})\mathbf{e}_2^t D = Q(\alpha\mathbf{j})(\mathbf{e}_1 + \mathbf{e}_2)^t D$$

has spanning column rank 1. Thus $\mathbf{z}\mathbf{e}_1^t + \alpha\mathbf{j}\mathbf{e}_2^t$ also has spanning column rank 1. Then $\mathbf{z} = r\alpha\mathbf{j}$ or $r\mathbf{z} = \alpha\mathbf{j}$ for some r But then $Q\mathbf{z} = \mathbf{o}$ and hence $T(\mathbf{z}\mathbf{e}_1^t) = \mathbf{o}$, contradicting that T preserves spanning column rank 1. Thus Q is nonsingular in $\mathbb{M}_{m,m}(\mathbf{R})$.□

COROLLARY 3.6. *Let $T : \mathbb{M}_{m,n}(\mathbf{U}_+) \to \mathbb{M}_{m,n}(\mathbf{U}_+)$ be a linear operator, $n \ge 2$. Then T preserves spanning column ranks if and only if there exist $Q \in \mathbb{M}_{m,m}(\mathbf{U}_+)$, D and $P \in \mathbb{M}_{n,n}(\mathbf{U}_+)$ such that $T(X) = QXDP$ for all $X \in \mathbb{M}_{m,n}(\mathbf{U}_+)$, where Q is invertible in $\mathbb{M}_{m,m}(\mathbf{R})$, D is an invertible diagonal matrix whose diagonal entries are all units in $\mathbf{U}_+$ and P is a permutation matrix.*

Now we have examples which show the Lemma 3.1, Theorem 3.5 and Corollary 3.6 do not hold if certain conditions were not satisfied.

EXAMPLE 3.7. Let $\mathbf{S} = (\mathbf{Z}[\sqrt{7}])_+$ and let $T : \mathbb{M}_{2,2}(\mathbf{S}) \to \mathbb{M}_{2,2}(\mathbf{S})$ be the linear operator given by $T(X) = X \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then T maps column matrices to column matrices and it is of the form QXDP. But the diagonal entries of D are not units over $\mathbf{S}$. So we show that T does not preserve spanning column ranks 1 and 2. Consider $X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with $sc(X_1) = 1$. Then $T(X_1) = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ has spanning column rank 2 over $\mathbf{S}$. And consider $X_2 = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$ with $sc(X_2) = 2$. Then $T(X_2) = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$ has spanning column rank 1 over $\mathbf{S}$. Moreover, this example shows that the converse of Lemma 3.1 does not hold.

EXAMPLE 3.8. Let $\mathbf{S} = \{0\} \cup \{q \in Q \mid q \geq 1\}$. Then $\mathbf{S}$ is a nonnegative semidomain, but $\mathbf{S} \neq \mathbf{U}_+$ for some unique factorization domain $\mathbf{U}$ in $\mathbf{R}$. Let $T : \mathbb{M}_{m,2}(\mathbf{S}) \to \mathbb{M}_{m,2}(\mathbf{S})$ be the linear operator given by $T(X) = X \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then we show that T preserves the spanning column rank of X. Certainly, T preserves spanning column rank 0. Let $X = [\mathbf{x}_1 \mid \mathbf{x}_2]$. Suppose that $sc(X) = 1$; then for some $q \in \mathbf{S}$, either $\mathbf{x}_1 = q\mathbf{x}_2$ or $\mathbf{x}_2 = q\mathbf{x}_1$. In the former case, we find that $T(X) = [(q+2)\mathbf{x}_2 \mid (2q+1)\mathbf{x}_2]$. If $q = 0$, then evidently $sc(T(X)) = 1$, while if $q \geq 1$, then $2q+1 \geq q+2$, and so $p = (2q+1)/(q+2) \in \mathbf{S}$. Since the second column of $T(X)$ is p times the first, we see that $T(X)$ has spanning column rank 1. A similar argument applies if $\mathbf{x}_2 = q\mathbf{x}_1$, and we see that T preserves spanning column rank 1

Conversely, suppose that $T(X) = [\mathbf{x}_1 + 2\mathbf{x}_2 \mid 2\mathbf{x}_1 + \mathbf{x}_2]$ has spanning column rank 1. Then for some $p \in \mathbf{S}$, either $\mathbf{x}_1 + 2\mathbf{x}_2 = p(2\mathbf{x}_1 + \mathbf{x}_2)$ or $2\mathbf{x}_1 + \mathbf{x}_2 = p(\mathbf{x}_1 + 2\mathbf{x}_2)$. Suppose that the former holds, and note that p cannot be 0 (otherwise $T(X)$ would be the zero matrix). Thus $p \geq 1$, and $(2-p)\mathbf{x}_2 = (2p-1)\mathbf{x}_1$. Note that $2 - p \geq 0$, and if $p = 2$, then $\mathbf{x}_1 = 0$, and it follows that $sc(X) = 1$. If $p < 2$, then from the fact that $2p - 1 \geq 2 - p$, we have $r = (2p-1)/(2-p) \in \mathbf{S}$. Since $\mathbf{x}_2 = r\mathbf{x}_1$, we see that $sc(X) = 1$. A similar argument applies if $2\mathbf{x}_1 + \mathbf{x}_2 = p(\mathbf{x}_1 + 2\mathbf{x}_2)$ and we see that if $sc(T(X)) = 1$ then $sc(X) = 1$. It now follows that T preserves spanning column ranks 1 and 2 on $\mathbb{M}_{m,2}(\mathbf{S})$.

But T does not map column matrices to column matrices since $T(E_{11}) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Moreover $T(X)$ is not of the form QXDP for any 2×2 invertible diagonal matrix D and a permutation matrix P over $\mathbf{S}$. □

We have studied the spanning column rank preservers on $\mathbb{M}_{m,n}(\mathbf{S})$ for $n \geq 2$. Now we make a remark on the spanning column rank preservers on $\mathbb{M}_{m,1}(\mathbf{S})$.

PROPOSITION 3.9. *Let $T : \mathbb{M}_{m,1}(\mathbf{S}) \to \mathbb{M}_{m,1}(\mathbf{S})$ be a linear operator. Then T preserves spanning column rank if and only if there exists $Q \in \mathbb{M}_{m,m}(\mathbf{S})$ each of whose columns have at least one nonzero entry such that $T(X) = QX$ for all $X \in \mathbb{M}_{m,1}(\mathbf{S})$.*

Proof. Let Q be the matrix of T with respect to the basis $\{E_{j1} \mid$

$j = 1, \ldots, m\}$ of $\mathbf{M}_{m,1}(\mathbf{S})$. Then $T(X) = QX$ and the jth column of Q is the coordinate of $T(E_{j1})$, which is not zero since T preserves spanning column rank 1. Hence each of the columns of Q has at least one nonzero entry. The converse is obvious.□

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Suk-Geun Hwang
Department of Mathematics Education
Teachers College, Kyungpook University
Taegu, 702-701, Republic of Korea

Si-Ju Kim
Department of Mathematics Education
Andong University
Andong, 760-749, Republic of Korea

Seok-Zun Song
Department of Mathematics
Cheju National University
Cheju, 690-756, Republic of Korea