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# SCALAR CURVATURES OF INVARIANT METRICS*

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## 1. Introduction

Let $(M, g)$ be a Riemannian manifold. The relation between a (pointwise) conformal metric of the metric $g$ and the scalar curvature of this new metric is investigated by Kazdan, Warner and Schoen (cf. [1, 4]).

In this note, we consider the scalar curvatures of Riemannian metrics of three dimensional sphere $S^3$. Since $S^3$ and $SU(2)$ are diffeomorphic, we deal with $SU(2)$ in place of $S^3$.

Throughout this paper, we investigate the relation between harmonic inner automorphisms of $(SU(2), g)$ and the scalar curvature for a given left invariant Riemannian metric (cf. Theorem 1). Moreover, we completely classify the scalar curvature of $(SU(2), g)$ for every left-invariant Riemannian metric $g$ into positive, negative, zero scalar curvatures (cf. Theorem 2).

## 2. Scalar curvatures of left invariant metrics on $SU(2)$

Let $M$ denote the Lie group $SU(2)$. We denote by $\mathfrak{g}$ the Lie algebra of all left invariant vector fields on $SU(2)$. The Killing form $B$ of $\mathfrak{g}$ satisfies $B(X, Y) = 4\ Trace(XY)$, $(X, Y \in \mathfrak{g})$. We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{g}$ by

$$\langle\ ,\ \rangle_0 := -B(X, Y), \quad (X, Y \in \mathfrak{g}). \tag{1}$$

$\langle\ ,\ \rangle_0$ determines a left invariant metric $g_0$ on $M$. The following Lemma is known (cf. [5, Lemma 1.1, p. 154]):

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LEMMA 1. *Let $g$ be an arbitrary left invariant Riemannian metric on $M$. Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{g}$ defined by $\langle X, Y\rangle := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{g}$ and $e$ is the identity matrix of $M$. Then there exists an orthonormal basis $(X_1, X_2, X_3)$ of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle_0$ such that*

$$\begin{cases} [X_1,\ X_2] &= (1/\sqrt{2})X_3, \qquad [X_2,\ X_3] = (1/\sqrt{2})X_1, \\ [X_3,\ X_1] &= (1/\sqrt{2})X_2, \qquad \langle X_i,\ X_j\rangle = \delta_{ij}\ a_i^2, \end{cases} \tag{2}$$

*where $a_i, (i = 1, 2, 3)$, are positive constant real numbers determined by the given left invariant Riemannian metric $g$ of $M$.*

We fix an orthonormal basis $(X_1, X_2, X_3)$ of $\mathfrak{g}$ with respect to $g_0$ with the property (2) in Lemma 1 and denote by $g(a_1, a_2, a_3)$, or simply by $g(a)$, the left invariant Riemannian metric on $M$ which is determined by positive real numbers $a_1, a_2, a_3$ in Lemma 1.

The connection function $\alpha$ on $\mathfrak{g}\times\mathfrak{g}$ corresponding to the left invariant Riemannian connection of $(M, g)$ is given as follows (cf. [2, p. 52]):

$$\begin{cases} \alpha(X,\ Y) &= 1/2\,[X,\ Y] + U(X,\ Y), \qquad (X, Y \in \mathfrak{g}), \\ \langle X, Y\rangle : &= g_e(X_e,\ Y_e), \end{cases} \tag{3}$$

where $U(X, Y)$ is determined by

$$\begin{aligned} &2\langle U(X,Y),\ Z\rangle \\ &= \langle [Z,X],\ Y\rangle + \langle X,\ [Z,Y]\rangle,\ (X,Y,Z \in \mathfrak{g}). \end{aligned} \tag{4}$$

Using (2) and (4), we get

LEMMA 2.

$$\begin{cases} U(X_1,\ X_1) &= U(X_2,\ X_2) = U(X_3,\ X_3) = 0, \\ U(X_1,\ X_2) &= (\sqrt{2}/4)a_3^{-2}({a_2}^2 - {a_1}^2)X_3, \\ U(X_2,\ X_3) &= (\sqrt{2}/4)a_1^{-2}({a_3}^2 - {a_2}^2)X_1, \\ U(X_3,\ X_1) &= (\sqrt{2}/4)a_2^{-2}({a_1}^2 - {a_3}^2)X_2, \end{cases} \tag{5}$$

*for the orthonormal basis $(X_1,\ X_2,\ X_3)$ with respect to $\langle\ ,\ \rangle_0$ and positive numbers $a_1, a_2, a_3$ in Lemma 1.*

$(X_1/a_1, X_2/a_2, X_3/a_3)$ is an orthonormal frame basis of $(M, g(a))$. Let $R$ (resp. $S(g(a))$) be the Ricci tensor (resp. the scalar curvature) of $(M, g(a))$. Then we have from (2), (3), and (5)

$$\begin{cases} R(Y_1, Y_1) &= (a_1^4 - a_2^4 + 2\, a_2^2 a_3^2 - a_3^4)/4c, \\ R(Y_2, Y_2) &= (-a_1^4 + a_2^4 + 2\, a_1^2 a_3^2 - a_3^4)/4c, \\ R(Y_3, Y_3) &= (-a_1^4 + 2\, a_1^2 a_2^2 - a_2^4 + a_3^4)/4c, \\ R(Y_1, Y_2) &= R(Y_2, Y_3) = R(Y_3, Y_1) = 0, \end{cases} \tag{6}$$

where $Y_i := X_i/a_i$, $i = 1, 2, 3$ and $c := a_1^2 a_2^2 a_3^2$. We get from (6)

$$S(g(a)) = (-a_1^4 + 2a_1^2 a_2^2 - a_2^4 + 2\, a_2^2 a_3^2 - a_3^4 + 2\, a_3^2 a_1^2)/4c. \tag{7}$$

The first author obtained the folowing result (cf. [3, Proposition 3.3]).

LEMMA 3. *An inner automorphism $A_x$, $(x = exp(rX_1),\ r \in R)$, of $(M, g(a))$ is harmonic if and only if*

$$a_2 = a_3 \quad or \quad r \in \{\, n\pi/\sqrt{2} \mid n \ \ is\ an\ integer\}. \tag{8}$$

Thus we get from (7) and Lemma 3

THEOREM 1. *Let $A_x$, $(x = exp(rX_1),\ r \in R)$, be an inner automorphism of $(M, g(a))$. Assume that $a_1^2 = 2(a_2^2 + a_3^2)$ and the scalar curvature of $(M, g(a))$ is zero. Then $A_x$ is a harmonic map.*

THEOREM 2. *Assume inner automorphisms $A_x$, $(x = exp(rX_1),\ r \notin \{n\pi/\sqrt{2} \mid n$ is an integer $\})$, of $(M, g(a))$ are harmonic maps. Then, the scalar curvature $S(g(a))$ of $(M, g(a))$ is as follows:*

*In the cases of $a_1^2 \leq 4\, a_2^2$, $a_1^2 = 4\, a_2^2$ and $a_1^2 \geq 4\, a_2^2$, $S(g(a))$ is positive, zero and negative respectively.*

REMARK. Similarly, the Theorems above can be stated for inner automorphisms $A_x$ of $(SU(2), g(a))$, $(x = exp(rX_2)$ or $exp(rX_3))$, (cf. [3, Proposition 3.4 and 3.5]).

## References

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