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A CHARACTERIZATION OF THE ALGEBRAIC MULTIPLICITY AS A MAP OF GROTHENDIECK GROUPS

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1. Introduction

The multiplicity theory initiated by C.Chevalley was the one with respect to ideals generated by a system of parameters of a local ring containing a field [3] and [4]. Samuel generalized the definition to primary ideals belonging the maximal ideal of a local ring which contains a field by a device which used the Hilbert characteristic function [9]. Furthermore Samuel defined multiplicity also in local rings which contain no field [10].

As a matter of fact, modern ideas in multiplicity theory sprang from applications of the theory of Hilbert functions to local rings. More recently, M.Auslander and D.A. Buchsbaum have taken up and extended an approach first suggested by J.P. Serre. They employed the methods of homological algebra to give an explicit expression for a general multiplicity in terms of the Euler Poincaré characteristic of the graded homology module of a certain Koszul complex. However, since the notion of algebraic multiplicity is of fundamental importance, there may be interest in an approach which uses neither Hilbert functions nor homological algebra.

D. J. Wright attempted an approach of this kind by a different road from the treatments which can be found in K. Blackburn [2], [13]. According to [13,p 269], D. Wright was inspired by theorem 3.3 of Auslander and Buchsbaum [1, 636]. He defined the general multiplicity as follows :

If R is a commutative Noetherian ring with unity, if $\gamma_1, \dots, \gamma_s$ is a sequence of elements of R satisfying the condition that $R/(\gamma_1, \dots, \gamma_s)$ has finite length, and E is a finitely generated R-module, then a symbol

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$e_R(\gamma_1, \dots, \gamma_s|E)$ is defined taking its values in the integers. A fundamental property of the symbol $e_R(\gamma_1, \dots, \gamma_s|E)$ is additivity on exact sequences; it is an Euler-Poincaré characteristic. M. Fraser generalized the general multiplicity theory without the finiteness of length of $R/(\gamma_1, \dots, \gamma_s)R$ as a homomorphism of the Grothendieck group of the category of finitely generated R-modules to the Grothendieck group of the category of finitely generated modules over $R/(\gamma_1, \dots, \gamma_s)R$. We will characterize this homomorphism.

2. Preliminaries

Throughout this paper R will be denote a commutative Noetherian ring with unity. By module we will always mean a finitely generated unitary R-module. Let $\gamma_1, \dots, \gamma_s \in R$ and let $E/(\gamma_1, \dots, \gamma_s)E$ have a finite length. Then we will say that$\gamma_1, \dots, \gamma_s$ is a multiplicity system on E.

From now on, $\gamma_1, \dots, \gamma_s$ will be a multiplicity system on each E and for each R-module E we require that $E/(\gamma_1, \dots, \gamma_s)E$ has finite length.

DEFINITION. : Notation as above. We put

$$\begin{aligned} &e_R(\gamma_1, \dots, \gamma_s|E) \\ =&e_{R/\gamma_1 R}(\gamma_2, \dots, \gamma_s|E/\gamma_1 E) - e_{R/\gamma_1 R}(\gamma_2, \dots, \gamma_s|0 :_E r_1) \end{aligned}$$

under the assumption that the symbols

$$e_{R/\gamma_1 R}(\gamma_2, \dots, \gamma_s|E/\gamma_1 E) \text{ and } e_{R/\gamma_1 R}(\gamma_2, \dots, \gamma_s|0 :_E \gamma_1)$$

are both defined, and we put

$$e_R(\cdot|E) = L_R(E) \text{ if } s = 0.$$

Such a symbol defined inductively will be called a multiplicity symbol and it will denoted by abbreviation $e_R(\gamma|E)$ if no confusion can arise.

Properties of e_R

i) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence and $\gamma_1, \ldots, \gamma_s$ a multiplicity system on each term, then

$$e_R(\gamma|E) = e_R(\gamma|E') + e_R(\gamma|E''), \text{ [13, p 271] }.$$

ii) Let $0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to 0$ be an exact sequence and $\gamma_1, \ldots, \gamma_s$ a multiplicity system on each term. Then

$$\sum_{i=0} (-1)^i e_R(\gamma|E_i) = 0.$$

By induction on s, it is easy to prove i) and ii) together.

iii) The multiplicity symbol has the exchange property :

$$e_R(\gamma|E) = e_R(\gamma_{i_1}, \ldots, \gamma_{i_s}|E),$$

where $\{i_1, \ldots, i_s\}$ is a permutation of $\{1, 2, \ldots, s\}$ [13, pp 274 – 275].

iv) Let E be a Noetherian R-module and $\gamma_1, \ldots, \gamma_s$ a multiplicity system on E.

Assume that for some particular value of i, $\gamma_i^m E = 0$, where m is a positive integer, then $e_R(\gamma|E) = 0$.

(Put $i = 1$ in iii) and use induction on m [13, p 272].)

v) $0 \le e_R(\gamma|E) \le L_R\{E/(\gamma)E\}$. Use $(0 :_{F_m} \gamma) = 0$, where $F_m = E/(0 :_E \gamma^m)$ for sufficiently large m [8, p 308].

COROLLARY. *If* $(\gamma_1, \ldots, \gamma_s)E = E$ *then* $e_R(\gamma|E) = 0$.

3. The Koszul complex

Let $\gamma_1, \ldots, \gamma_s \in R$. The Koszul complex of E with respect to $\gamma_1, \ldots, \gamma_s$ will be denoted by $K(\gamma_1, \ldots, \gamma_s|E)$ and it will be abbreviated by $K(\gamma|E)$ if no confusion can arise. Our main tool is the Koszul complex in the same style as [8].

Properties of the Koszul complex:

i) Let $0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$ be an exact sequence of R-modules and $\gamma_1, \cdots, \gamma_s \in R$. Then we have ;

$$\begin{aligned} &0 \to H_sK(\gamma|E') \to H_sK(\gamma|E) \to H_sK(\gamma|E'') \to \cdots \\ &\to H_\mu K(\gamma|E') \to H_\mu K(\gamma|E) \to H_\mu K(\gamma|E'') \to \cdots \\ &\to H_0K(\gamma|E') \to H_0K(\gamma|E) \to H_0K(\gamma|E'') \to 0 \text{ [8, pp 362-363]} . \end{aligned}$$

ii) Let $A = (\gamma_1, \ldots, \gamma_s)R$. Then A annihilates $H_\mu K(\gamma_1, \ldots, \gamma_s|E)$ for all μ. Here, $H_\mu K(\gamma_1, \ldots, \gamma_s|E)$ are the homology modules with respect to $\gamma_1, \cdots, \gamma_s$ [8, p 364].

iii) The homology modules of the Koszul complexes $K(\gamma_1, \ldots, \gamma_{s-1}|E)$ and $K(\gamma_1, \ldots, \gamma_s|E)$ are connected through an exact sequence [8, p 365].

$$\begin{array}{llll}
\longrightarrow & H_{\mu+1}K(\gamma_1, \ldots, \gamma_{s-1}|E) & \longrightarrow & H_{\mu+1}K(\gamma_1, \ldots, \gamma_s|E) \\
\longrightarrow & H_\mu K(\gamma_1, \ldots, \gamma_s|E) & \xrightarrow{(-1)^\mu \gamma_s} & H_\mu K(\gamma_1, \ldots, \gamma_{s-1}|E) \\
\longrightarrow & H_\mu K(\gamma_1, \ldots, \gamma_s|E) & \longrightarrow & H_{\mu-1}K(\gamma_1, \ldots, \gamma_{s-1}|E) \\
\xrightarrow{(-1)^{\mu-1}\gamma_s} & H_{\mu-1}K(\gamma_1, \ldots, \gamma_{s-1}|E) & \longrightarrow & H_{\mu-1}K(\gamma_1, \ldots, \gamma_s|E) \\
\longrightarrow & H_{\mu-2}K(\gamma_1, \ldots, \gamma_{s-1}|E) & \longrightarrow & \cdots\cdots \\
\xrightarrow{(-1)\gamma_s} & H_1K(\gamma_1, \ldots, \gamma_{s-1}|E) & \longrightarrow & H_1K(\gamma_1, \ldots, \gamma_s|E) \\
\longrightarrow & H_0K(\gamma_1, \ldots, \gamma_{s-1}|E) & \xrightarrow{(-1)^0\gamma_s} & H_0K(\gamma_1, \ldots, \gamma_{s-1}|E) \\
\longrightarrow & H_0K(\gamma_1, \ldots, \gamma_s|E) & \longrightarrow & 0
\end{array}$$

iv) If $(0 :_E \gamma_s) = 0$, we have

$$H_\mu K(\gamma_1, \ldots, \gamma_s|E) \simeq H_\mu K(\gamma_1, \ldots, \gamma_{s-1}|E/\gamma_s E) \text{ [8, p 368]}$$

THEOREM. *Put* $\mathcal{X}_R(\gamma_1, \ldots, \gamma_s|E) = \sum_{\mu=0}(-1)^\mu L_R\{H_\mu K(\gamma_1, \ldots, \gamma_s)|E)\}$, *then*

$$e_R(\gamma_1, \ldots, \gamma_s|E) = \mathcal{X}_R(\gamma_1, \ldots, \gamma_s|E) \text{[8, p 369]} .$$

To prove this, we need two Lemmas :

i) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of Noetherian R-modules and suppose that $\gamma_1, \dots, \gamma_s$ is a multiplicity system for each of them. Then

$$\mathcal{X}_R(\gamma_1, \dots, \gamma_s|E) = \mathcal{X}_R(\gamma_1, \dots, \gamma_s|E') + \mathcal{X}_R(\gamma_1, \dots, \gamma_s|E'') \text{ [8, p 370].}$$

ii) Let $E, \gamma_1, \dots, \gamma_s$ be as above. If $\gamma_s^m E = 0$, where $m > 0$, then

$$\mathcal{X}_R(\gamma_1, \dots \gamma_s|E) = 0. \text{ [8, p 370]}$$

PROOF OF THEOREM. *We use induction on s. The theorem is clear in the case of $s = 0$. Suppose that $s > 0$ and the theorem has been established for multiplicity systems with $s - 1$ elements. Put $F = E/(0 :_E \gamma_s^m)$, where $m \gg 0$ so that $(0 :_F \gamma_s) = 0$. (See 2, V)) Then applying i) above to the exact sequence*

$$0 \to (0 :_E \gamma_s^m) \to E \to F \to 0,$$

$$\text{we get } \mathcal{X}_R(\gamma|E) = \mathcal{X}_R(\gamma|F) + \mathcal{X}_R(\gamma|0 :_E \gamma_s^m).$$

By ii) above, the last term equals zero. Since γ is not zero divisor on F, by property iv) of the Koszul complex

$$H_\mu K(\gamma_1, \dots, \gamma_s|F) \simeq H_\mu K(\gamma_1, \dots, \gamma_{s-1}|F/\gamma_s F)$$

$$\text{so } \mathcal{X}_R(\gamma_1, \dots, \gamma_s|F) = \mathcal{X}_R(\gamma_1, \dots, \gamma_s|F/\gamma_s F) = \mathcal{X}_R(\gamma_1, \dots, \gamma_s|E)$$

On the other hand, we have

$$e_R(\gamma_1, \dots, \gamma_s|E) = e_R(\gamma_1, \dots, \gamma_s|F) = e_R(\gamma_1, \dots, \gamma_s|F/\gamma_s F).$$

For, from the exact sequence

$$0 \to (0 :_E \gamma^m) \to E \to E/(0 :_E \gamma^m) \to 0,$$

we get an equation ;

$$\begin{aligned} &e_R(\gamma_1, \dots, \gamma_s|E) \\ &= e_R(\gamma_1, \dots, \gamma_s|F) + e_R(\gamma_1, \dots, \gamma_s|(0 :_E \gamma^m)) \text{ by 2, i).} \end{aligned}$$

The last term equals zero by property 2, iv) of the multiplicity symbol since

$$r^m(0 :_E \gamma^m) = 0.$$

Moreover

$$e_R(\gamma_1, \dots, \gamma_s|F) = e_R(\gamma_1, \dots, \gamma_{s-1}|F/r_s F)$$

by a similar argument as used in basic property v) of the multiplicity symbol since $(0 :_F \gamma_s) = 0$. *The last term of the above equation equals* $\mathcal{X}_R(\gamma_1, \dots, \gamma_{s-1}|F/\gamma_s F)$ *by the induction hypothesis. Therefore,*

$$\mathcal{X}_R(\gamma_1, \dots, \gamma_{s-1}|F/\gamma_s F) = \mathcal{X}_R(\gamma_1, \dots, \gamma_s|F) = \mathcal{X}_R(\gamma_1, \dots, \gamma_s|E).$$

This is the reguired result.

4. Algebraic multiplicity as a map of the Grothendieck group of some abelian category

Let $\mathcal{C}$ be an abelian category and $\mathcal{K}^0(\mathcal{C})$ its Grothendieck group of $\mathcal{C}$.
In the special case of the category of all finitely generated R-modules its Grothendieck group will be denoted by $\mathcal{K}(R)$ and $[E]$ will denote the class of E in $\mathcal{K}^0(\mathcal{C})$ for $E \in \mathcal{C}$.
Let $\gamma_1, \dots, \gamma_s \in R$ and I be the ideal generated by $\gamma_1, \dots, \gamma_s$. Define a mapping $\mathcal{X}_R(\gamma_1, \dots, \gamma_s) : \mathcal{K}(R) \to \mathcal{K}(R/I)$ by

$$\mathcal{X}_R(\gamma_1, \dots, \gamma_s)[E] = \sum_{i=0}^{s} (-1)^i [H_i K(\gamma_1, \dots, \gamma_s|E)] \text{ and if } s = 0, \text{ put } \mathcal{X}_R(\cdot)[E] = [E]$$

i.e., identity map. Recall $H_i K(\gamma_1, \dots, \gamma_s|E) = 0$ if $i < 0$ and $i > s$, and $IH_i K(\gamma_1, \dots, \gamma_s|E) = 0$ for all $0 \le i \le s$ (see 3, ii)) and therefore $H_i K(\gamma_1, \dots, \gamma_s|E)$ can be regarded as an R/I-module.

Properties of the map $\mathcal{X}_R(\gamma_1, \dots, \gamma_s)$

i) Let $I = (\gamma_1, \dots, \gamma_s)$ and $0 \to E' \to E \to E'' \to 0$ be an exact sequence of $\mathcal{C}$ then

$$\mathcal{X}_R(\gamma_1, \dots, \gamma_s)[E] = \mathcal{X}_R(\gamma_1, \dots, \gamma_s)[E'] + \mathcal{X}_R(\gamma_1, \dots, \gamma_s)[E''].$$

Proof. Split up the exact sequence obtained in i) of 3 into short exact sequences:

$$
\begin{array}{ccccccccccc}
 & & & H_sK(\gamma|E') & & \longrightarrow & & H_sK(\gamma|E) & & & \\
 & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
 & 0 & & & & \operatorname{im}\psi_s & & & & \operatorname{im}\phi_s & \\
 \nearrow & & & & \nearrow & & \searrow & & & & \searrow \\
0 & & & & 0 & & 0 & & & & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & \longrightarrow H_sK(\gamma|E'') & & \longrightarrow & & H_{s-1}K(\gamma_1,\cdots,\gamma_{s-1}\ |E') & & & \\
 & \nearrow & & \searrow & & \nearrow & \searrow & & \\
\operatorname{im}\phi_s & & & & \Delta_s & & & \operatorname{im}\psi_s & \\
\nearrow & & & \nearrow & & \searrow & & & \searrow \\
0 & & & 0 & & 0 & & & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
\longrightarrow & H_{s-1}\ K(\gamma_1,\cdots,\gamma_{s-1}\ |E) & & & \longrightarrow & H_{s-1}K(\gamma_1,\cdots,\gamma_{s-1}\ |E'') & & & \\
 & \nearrow & & \searrow & & \nearrow & \searrow & & \\
\operatorname{im}\psi_{s-1} & & & \operatorname{im}\phi_{s-1} & & & & \Delta_{s-1} & \\
\nearrow & & & \nearrow & & \searrow & & & \searrow \\
0 & & & 0 & & 0 & & & 0
\end{array}
$$

$\longrightarrow$ ....

Then,

$$
\begin{array}{ll}
i+1>s \Longrightarrow \Delta_{i+1}=0 & [H_iK(\gamma|E')] = [\text{ im } \Delta_{i+1}] + [\text{ im } \psi_i] \\
0 \le i \le s, & [H_iK(\gamma|E)] = [\text{ im } \psi_i] + [\text{ im } \phi_i] \\
 & [H_iK(\gamma|E'')] = [\text{ im } \phi_i] + [\text{ im } \Delta_i]
\end{array}
$$

From these, we have

$$[H_iK(\gamma|E)] = [H_iK(\gamma|E')] - [\operatorname{im}\Delta_{i+1}] + [H_iK(\gamma|E'')] - [\operatorname{im}\Delta_i]$$

$$\text{But } [\text{ im } \Delta_{i+1}] + [\text{ im } \Delta_i] = 0.$$

Therefore,

$$\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E] = \mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E'] + \mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E''].$$

ii) If $\gamma_1^n E = 0$ for some n then

$$\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E] = 0.$$

We replace γ_1 by γ_s by the exchange property 2, iii). From the basic property 3, iii) looking at three terms

$$\to H_i K(I|E) \to H_{i-1}K(I|E) \xrightarrow{\gamma_s} H_{i-1}K(I_{\gamma_s}|E) \to ,$$

we have that

$$\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E] = \mathcal{X}_R(\gamma_1,\ldots,\gamma_{s-1})(E)+\mathcal{X}_R(\gamma_1,\ldots,\gamma_{s-1})(E) = 0$$

iii) If γ_1 is not zero divisor on E. Then

$$\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)[E] = \mathcal{X}_R(\gamma_2,\ldots,\gamma_s)[E/r_1E]$$

Proof. Use the basic property 3, iv).

5. The Main theorem

THEOREM. *Let R be a commutative noetherian ring with unity and $\mathcal{C}$ be the category of all finitely generated unitary R-modules. Choose $\gamma_1,\ldots,\gamma_i,\ldots,\gamma_s \in R$ and let I_{γ_i} be the ideal generated by $\{\gamma_1,\ldots,\widehat{\gamma}_i,\ldots,\gamma_s\}$ with $\widehat{\gamma}_i$ deleted.*

Suppose that for each pair (I,E) for $E \in \mathcal{C}$, $\psi(I)$ is a map $\mathcal{K}(R) \to \mathcal{K}(R/I)$ satisfying the followings :

1) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence in $\mathcal{C}_R$, then

$$\psi(I)[E] = \psi(I)[E'] + \psi(I)[E''].$$

2) If $\gamma_i^m E = 0$ for some integers $m > 0$, where γ_i is one of $\gamma_1,\ldots,\gamma_s$, then

$$\psi(I)[E] = 0$$

3) If $(0 :_E \gamma) = 0$, then $\psi(I)[E] = \psi(I_{\gamma_i})[E/\gamma_1 E]$.

4) $\psi(\cdot)[E] = [E]$.

Then $\mathcal{X}_R(\gamma_1,\ldots,\gamma_s) = \psi(I)$.

Proof. That $\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)$ satisfies 1) - 4) has been already established. Conversely we shall prove that the above map must be $\mathcal{X}_R(\gamma_1,\ldots,\gamma_s)$. To begin with we recall the statement which was used in 1. Let E be a Noethenian R-module and $\gamma \in R$. Put $F = E/(0 :_E \gamma^m)$. Then $(0 :_F \gamma) = 0$ provided that m is sufficiently large. We use induction on s. When $s = 0$, the theorem is true by condition 4). It will therefore be supposed that $s > 0$ and that the thoerem has been established for $s - 1$ elements. Put $F = E/(0 :_E \gamma_s^m)$, where m is chosen large enough to ensure that γ_s is not a zero divisor on F. This is possible by 2, v). Then by applying condition 1) to the exact sequence

$$0 \to 0 :_E \gamma_s^m \to E \to F \to 0,$$

we have $\psi(I)[E] = \psi(I)[0 :_E \gamma_s^m] + \psi(I)[F]$,

which by condition 2 reduces to $\psi(I)[E] = \psi(I)[F]$.

Furthermore since $(0 :_F \gamma_s) = 0$, by condition 3)

$$\psi(I)[F] = \psi(I/(\gamma_s))[F/\gamma_s F] = \mathcal{X}_R(\gamma_1, \ldots, \gamma_{s-1})[F/r_s F].$$

Since $H_i K(\gamma_1, \ldots, \gamma_s | F) \simeq H_i K(\gamma_1, \ldots, \gamma_{s-1} | F/\gamma_s F)$,

by the basic property of the Koszul complex it follows that

$$\mathcal{X}_R(\gamma_1, \ldots, \gamma_s)[F] = \mathcal{X}_R(\gamma_1, \ldots, \gamma_{s-1})[F/\gamma_s F]$$

and therefore $\mathcal{X}_R(\gamma_1, \ldots, \gamma_s)[E] = \mathcal{X}_R(\gamma_1, \ldots, \gamma_{s-1})[F/\gamma_s F] = \psi(I)$ $[E]$. Thus the theorem is proved.

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