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ON THE EXTENDED JIANG SUBGROUP

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1.Introduction

F. Rhodes [2] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space X and showed a sufficient condition for $\sigma(X, x_0, G)$ to be isomorphic to $\pi_1(X, x_0) \times G$, that is, if (G, G) admits a family of preferred paths at e, $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$. B.J.Jiang [1] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group of X which depends on f and showed a condition to be $J(f, x_0) = Z(f_\pi(\pi_1(X, x_0)), \pi_1(X, f(x_0)))$. The author and Han [4] introduced the extended Jiang subgroup $J(f, x_0, G)$ of the fundamental group of a transformation group as an extension of the Jiang subgroup $J(f, x_0)$. In this paper, we want to show a condition to be $J(f, x_0, G) = Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$.

2. Definitions and Notations

Let (X, G, π) be a transformation group and X be a path connected compact ANR with x_0 as base point. Given an element g of G, *a path α of order g* with base point x_0 is a continuous map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$. A path α_1 of order g_1 and a path α_2 of order g_2 give rise to a path $\alpha_1 + g_1\alpha_2$ of order g_1g_2 defined by the equations

$$(\alpha_1 + g_1\alpha_2)(s) = \begin{cases} \alpha_1(2s), & 0 \leq s \leq 1/2 \\ g_1\alpha_2(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

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Two paths α and α' of the same order g are said to be *homotopic* if there is a continuous map $F : I^2 \longrightarrow X$ such that

$$F(s,0) = \alpha(s) \qquad 0 \le s \le 1,$$
$$F(s,1) = \alpha'(s) \qquad 0 \le s \le 1,$$
$$F(0,t) = x_0 \qquad 0 \le t \le 1,$$
$$F(1,t) = gx_0 \qquad 0 \le t \le 1.$$

The homotopy class of a path α of order g is denoted by $[\alpha; g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F. Rhodes[2] showed that the set of homotopy classes of paths of prescribed order with the rule of composition $\circ$ is a group, where $\circ$ is defined by $[\alpha_1; g_1] \circ [\alpha_2; g_2] = [\alpha_1 + g_1\alpha_2; g_1g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the *fundamental group* of (X, G) with base point x_0.

Let f be a self map of X. A homotopy $H : X \times I \longrightarrow X$ is said to be an *f-cyclic homotopy* if $H(\cdot, 0) = f = H(\cdot, 1)$. In this case, the path $H(x_0, \cdot)$ is called the traces of the f-cyclic homotopy H at x_0. In [1], Jiang has defined $J(f, x_0) = \{[\alpha] \in \pi_1(X, f(x_0)) | \alpha$ is homotopic to the traces of an f-cyclic homotopy at $x_0\}$. An equivalent definition of $J(f, x_0)$ is the following: Define $p : X^X \longrightarrow X$ by $p(g) = g(x_0)$; then p induces a homomorphism $p_\pi : \pi_1(X^X, f) \longrightarrow \pi_1(X, f(x_0))$. The Jiang subgroup $J(f, x_0)$ is the image of the homomorphism p_π. A homotopy $H : X \times I \longrightarrow X$ is said to be an *f-cyclic homotopy of order g* if $H(\cdot, 0) = f$ and $H(\cdot, 1) = gf$,where g is an element of G.

Definition 1. $J(f, x_0, G) = \{[\alpha; g] \in \sigma(X, f(x_0), G) | \alpha$ is homotopic to the traces of an f-cyclic homotopy of order $g\}$.[4]

If we define $i_G : J(f, x_0) \longrightarrow J(f, x_0, G)$ by $i_G([\alpha]) = [\alpha : e]$, then the Jiang subgroup $J(f, x_0)$ is identified with a subgroup of $J(f, x_0, G)$. Thus $J(f, x_0, G)$ is called the extended Jiang subgroup. Define $\pi' : X^X \times G \longrightarrow X^X$ by $\pi'(f, g) = gf$,then (X^X, G, π') is a transformation group and $p : (X^X, G) \longrightarrow (X, G)$ is a homomorphism. Thus p induces a homomorphism $p_\sigma : \sigma(X^X, f, G) \longrightarrow \sigma(X, f(x_0), G)$ given by $p_\sigma([\alpha : g]) = [p\alpha : g]$. It is easy to show that $p_\sigma(\sigma(X^X, f, G)) = J(f, x_0, G)$.

In [2], a transformation group (X, G) is said to admit a *family of preferred paths* at x_0 if it is possible to associate with every element g

of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is homotopic to $\hat{x}_0$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$, where $\hat{x_0}(t) = x_0$ for each $t \in I$.

3. An extension of the Jiang's results

In [2], Rhodes has shown that if (G, G) admits a family of preferred paths at e, then $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$. Now we look for a condition to be $J(f, x_0, G) = Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$.

DEFINITION 2. A family $\mathbf{K}$ of preferred paths at $f(x_0)$ is called a *family of preferred f-traces at x_0* if for every preferred path k_g in $\mathbf{K}$, $k_g\rho$ is the traces of an f-cyclic homotopy of order g at x_0, where $\rho(t) = 1 - t$. Especially, a family $\mathbf{K}$ of preferred 1_X-traces at $f(x_0)$ is a family of preferred traces at $f(x_0)$ which was defined in [3].

THEOREM 1. *Let (X, G, π) be a transformation group. If (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred f-traces at x_0 for any self map f of X.*

Proof. Let $\mathbf{H}$ be a family of preferred paths at e in (G, G). Define $\mathbf{K} = \{k_g | k_g(t) = \pi(f(x_0), h_g(t)), h_g \in \mathbf{H}\}$. Let $F : X \times I \longrightarrow X$ be a map such that

$$F(x, t) = \pi(f(x), h_g\rho(t))$$

Then

$$F(x, 0) = \pi(f(x), h_g(1)) = h_g(1)f(x) = f(x),$$

$$F(x, 1) = \pi(f(x), h_g(0)) = h_g(0)f(x) = gf(x)$$

and

$$F(x_0, t) = \pi(f(x_0), h_g\rho(t)) = h_g\rho(t)f(x_0) = k_g\rho(t).$$

Thus F is an f-cyclic homotopy of order g with trace $k_g\rho$. Therefore $\mathbf{K}$ is a family of preferred f-traces at x_0.

The following example shows that the converse of Theorem 1 does not hold.

EXAMPLE. Let R be the real space, Z the additive integer group and $\pi : R \times Z \longrightarrow R$ a map defined by $\pi(r, n) = r + n$. Then (R, Z, π) is a transformation group and it admits a family of preferred f-traces at 0 for any self map f of R. Because, let $\mathbf{K} = \{k_n | k_n$ is a path from n to 0 in $R\}$,then $\mathbf{H} = \{h_n | h_n = f(0) + k_n, k_n \in K\}$ is a family of preferred paths at $f(0)$. For each $n \in Z$, define $G : R \times I \longrightarrow R$ by $G(r, t) = f(r) + k_n\rho(t)$. Then G is an f-cyclic homotopy of order n with trace $h_n\rho$. Thus $\mathbf{H}$ is a family of preferred f-traces at 0. Since Z is a discrete space, there is no path from n to 0 in Z, where n is any nonzero integer. Thus (Z, Z) cannot admit a family of preferred paths at 0.

THEOREM 2. *Let f be a self map of X. Then the existence of a family of preferred f-traces is independent of the representatives of the homotopy class of f.*

Proof. Let f, f' be homotopic self maps of X and H be a homotopy from f to f'. We assume that f admits a family $\mathbf{K} = \{k_g | g \in G\}$ of preferred f-traces at x_0. Let α be the traces of the homotopy H at x_0, that is, $\alpha(t) = H(x_0, t)$ for each $t \in I$. Define $\mathbf{K}' = \{h_g | h_g = g\alpha\rho + k_g + \alpha, k_g \in K\}$. Since,

$$h_e = e\alpha\rho + k_e + \alpha \sim \hat{f'(x_0)}$$

and

$$\begin{aligned} h_{g_1 g_2} &= g_1 g_2 \alpha\rho + k_{g_1 g_2} + \alpha \\ &\sim g_1 g_2 \alpha\rho + (g_1 k_{g_2} + k_{g_1}) + \alpha \\ &\sim g_1(g_2\alpha\rho + k_{g_2} + \alpha) + (g_1\alpha\rho + kg_1 + \alpha) \\ &\sim g_1 h_{g_2} + h_{g_1}, \end{aligned}$$

$\mathbf{K}'$ is a family of preferred paths at $f'(x_0)$,where $h \sim h'$ denotes h is homotopic to h'. Next, we show that $\mathbf{K}'$ is a family of preferred f'-traces at x_0. Since $\mathbf{K}$ is a family of preferred f-traces at x_0, there exists an f-cyclic homotopy J of order g with trace $k_g\rho$ for each $g \in G$. Let us define a homotopy $J' : X \times I \longrightarrow X$ by

$$J'(x, t) = \begin{cases} H(x, 1 - 3t), & 0 \le t \le 1/3 \\ J(x, 3t - 1), & 1/3 \le t \le 2/3 \\ gH(x, 3t - 2), & 2/3 \le t \le 1 \end{cases}$$

Then $J'(x,0)=f'(x)$, $J'(x,1)=gf'(x)$ and $J'(x_0,\cdot)=\alpha\rho+k_g\rho+g\alpha=(g\alpha\rho+k_g+\alpha)\rho=h_g\rho$. Thus $h_g\rho$ is the traces of an f'-cyclic homotopy J' of order g at x_0.

THEOREM 3. *Let (X,G) be a transformation group. If λ is a path from x_0 to x_1, then a family of preferred f-traces at x_0 gives rise to a family of preferred f-traces at x_1.*

Proof. Let $\mathbf{K}=\{k_g|g\in G\}$ be a family of preferred f-traces at x_0. For each element g of G, let $h_g=gf\lambda\rho+k_g+f\lambda$. Since

$$h_e=f\lambda\rho+k_e+f\lambda\sim \hat{f}(x_1)$$

and

$$\begin{aligned}
h_{g_1g_2}&=(g_1g_2)f\lambda\rho+k_{g_1g_2}+f\lambda\\
&\sim(g_1g_2)f\lambda\rho+g_1k_{g_2}+k_{g_1}+f\lambda\\
&\sim(g_1g_2)f\lambda\rho+g_1k_{g_2}+g_1f\lambda+g_1f\lambda\rho+k_{g_1}+f\lambda\\
&\sim g_1(g_2f\lambda\rho+k_{g_2}+f\lambda)+(g_1f\lambda\rho+k_{g_1}+f\lambda)\\
&\sim g_1h_{g_2}+h_{g_1},
\end{aligned}$$

$\mathbf{H}=\{h_g|g\in G\}$ is a family of preferred paths at x_1. Since the induced isomorphism $(f\lambda)_*$ carries $J(f,x_0,G)$ isomorphically onto $J(f,x_1,G)$ by Theorem 8 in [4], $(f\lambda)_*[k_g\rho:g]=[f\lambda\rho+k_g\rho+gf\lambda:g]=[h_g\rho:g]$ belongs to $J(f,x_1,G)$ for any element $[k_g\rho\ \ g]$ of $J(f,x_0,G)$. Thus $\mathbf{H}=\{h_g|g\in G\}$ is a family of preferred f-traces at x_1.

LEMMA 4. *Let $f:X\longrightarrow X$ be a self map. If k is the traces of an f-cyclic homotopy of order g at x_0, then for every loop α at x_0, $f\alpha$ is homotopic to $k+gf\alpha+k\rho$. In particular, if f is a homeomorphism and α is a loop at $f(x_0)$, α is homotopic to $k+g\alpha+k\rho$*

Proof. Let $H:X\times I\longrightarrow X$ be an f-cyclic homotopy of order g with trace k and α be a loop at x_0. Define $F:I\times I\longrightarrow X$ by

$$F(s,t)=\begin{cases} k(4s), & 0\le s\le t/4\\ H(\alpha((4s-t)/(4-2t)),t), & t/4\le s\le(4-t)/4\\ k\rho(4s-3), & (4-t)/4\le s\le 1.\end{cases}$$

Then F is well defined, $F(s,0) = H(\alpha(s),0) = (f\alpha)(s)$ and $F(s,1) = (k+gf\alpha+k\rho)(s)$. In particular, suppose that f is a homeomorphism and β is a loop at $f(x_0)$. Then $f^{-1}\beta$ is a loop at x_0. Thus β is homotopic to $k + gf(f^{-1}\beta) + k\rho = k + g\beta + k\rho$.

For a group G and a subgroup H, the centralizer $Z(H,G)$ of H in G is the subgroup of G defined by $Z(H,G) = \{g \in G | gh = hg$ for all $h \in H\}$. In [1], Jiang has shown that $J(f,x_0) \subset Z(f_\pi(\pi_1(X,x_0)), \pi_1(X, f(x_0)))$. Now we generalize this result to the fundamental group of a transformation group as the following:

THEOREM 5. *Let f be an endomorphism of (X,G) and G be abelian. If (X,G) admits a family $\{k_g | g \in G\}$ of preferred 1_X-traces at $f(x_0)$, where k_g is homotopic to an f-image of a path from gx_0 to x_0, then $J(f,x_0,G) \subset Z(f_\sigma(\sigma(X,x_0,G)), \sigma(X,f(x_0),G))$.*

Proof. Let $\mathbf{K} = \{k_g | g \in G\}$ be a family of preferred 1_X-traces at $f(x_0)$,where k_g is homotopic to an f-image of a path h_g from gx_0 to x_0. For $[\alpha : g_1] \in J(f,x_0,G)$ and $[\beta : g_2] \in \sigma(X,x_0,G)$, we must show $[\alpha : g_1] \circ f_\sigma[\beta : g_2] = f_\sigma[\beta : g_2] \circ [\alpha : g_1]$. Since G is abelian, it is sufficient to show that $\alpha + g_1 f\beta$ is homotopic to $f\beta + g_2\alpha$. If we use Lemma 4 and $k_g\, \rho$ is the traces of 1_X-cyclic homotopy of order g_1 at $f(x_0)$. we have

$$\begin{aligned}
\alpha + g_1 f\beta &\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1 f\beta + k_{g_1 g_2} + k_{g_1 g_2}\rho \\
&\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1(f\beta + k_{g_2}) + k_{g} + k_{g_1 g_2}\rho \\
&\sim \alpha + k_{g_1} + f\beta + k_{g_2} + k_{g_1 g_2}\rho
\end{aligned}$$

and

$$\begin{aligned}
f\beta + g_2\alpha &\sim f\beta + k_{g_2} + k_{g_2}\rho + g_2\alpha + k_{g_2 g_1} + k_{g_2 g_1}\rho \\
&\sim f\beta + k_{g_2} + k_{g_2}\rho + g_2(\alpha + k_{g_1}) + k_{g_2} + k_{g_2 g_1}\rho \\
&\sim f\beta + k_{g_2} + \alpha + k_{g_1} + k_{g_2 g_1}\rho.
\end{aligned}$$

From these results, we know that $\alpha + g_1 f\beta$ is homotopic to $f\beta + g_2\alpha$ if and ony if $\alpha + k_{g_1} + f\beta + k_{g_2}$ is homotopic to $f\beta + k_{g_2} + \alpha + k_{g_1}$. Since $[\alpha : g_1] \in J(f, x_0, G)$ and $k_{g_1} \in \mathbf{K}$, there exists an f-cyclic homotopy H of order g_1 at x_0 such that $H(x_0, \cdot)$ is homotopic to α and 1_X-cyclic homotopy G of order g_1 at $f(x_0)$ such that $G(f(x_0),\cdot)$ is homotopic to $k_{g_1}\rho$. Define $J : X \times I \longrightarrow X$ by

$$J(x,t) = \begin{cases} H(x,2t), & 0 \leq t \leq 1/2 \\ G(f(x), 2-2t), & 1/2 \leq t \leq 1. \end{cases}$$

then J is an f-cyclic homotopy such that $J(x_0,\cdot)$ is homotopic to $\alpha + k_{g_1}$. Thus $\alpha + k_{g_1}$ is the traces of an f-cyclic homotopy of order 1_X. By Lemma 4 and $k_{g_2} \sim fh_{g_2}$ for a path h_{g_2} from g_2x_0 to x_0, we obtain

$$\begin{aligned} \alpha + k_{g_1} + f\beta + k_{g_2} &\sim \alpha + k_{g_1} + f\beta + fh_{g_2} \\ &\sim (\alpha + k_{g_1}) + (\alpha + k_{g_1})\rho - f(\beta + h_{g_2}) + (\alpha + k_{g_1}) \\ &\sim f\beta + k_{g_2} + \alpha + k_{g_1}. \end{aligned}$$

Let G be the trivial group $\{1_X\}$. Then G is abelian, any self-map of X is an endomorphism of (X,G) and (X,G) has a family K of preferred 1_X-traces at $f(x_0)$, where the only one element of K is the f-image of the constant path at x_0. As a corollary of the above theorem, we have the following Jiang's result in [1].

COROLLARY 6. *If f is a self-map of X, then $J(f,x_0) \subset Z(f_\pi(\pi_1(X, x_0)), \pi_1(X, f(x_0)))$.*

COROLLARY 7. *Let f be an automorphism of (X,G) and G be abelian. If (X,G) admits a family $\{k_g | g \in G\}$ of preferred 1_X-traces at $f(x_0)$, then $J(f,x_0,G)$ is contained in $Z(\sigma(X, f(x_0), G))$.*

In [1], the main result concerning the Jiang subgroup of maps on connected aspherical (in the sence that $\pi_i(X,x_0) = 0$ for $i > 1$) polyhedron is that if X is a connected aspherical polyhedron and f is a self map of X, we have $J(f,x_0) = Z(f_\pi\pi_1(X,x_0), \pi_1(X, f(x_0)))$.

Let X be a connected aspherical polyhedron and f be an endomorphism of (X,G). If (X,G) admits a family $\{k_g | g \in G\}$ of preferred 1_X-traces at $f(x_0)$, where k_g is homotopic to an f-image of a path from gx_0 to x_0, then the extended Jiang subgroup $J(f,x_0,G)$ can be given explicitly.

THEOREM 8. *Let X be a connected aspherical polyhedron, G be abelian and f be an endomorphism of (X,G). If (X,G) admits a family $\{k_g|g \in G\}$ of preferred 1_X-traces at $f(x_0)$ where k_g is homotopic to an f-image of a path from gx_0 to x_0, then $J(f,x_0,G) = Z(f_\sigma(\sigma(X,x_0,G)),\sigma(X,f(x_0),G))$.*

Proof. Let f be an endomorphism of (X,G). If (X,G) admits a family $\{k_g|g \in G\}$ of preferred 1_X-traces at $f(x_0)$,where k_g is homotopic to an f-image of a path h_g from gx_0 to x_0, then we first show that there exists an isomorphism ϕ from $\sigma(X,f(x_0),G)$ onto $\pi_1(X,f(x_0)) \times G$ which carries $J(f,x_0,G)$ onto $J(f,x_0) \times G$. Define $\phi : \sigma(X,f(x_0),G) \longrightarrow \pi_1(X,f(x_0)) \times G$ by $\phi([\alpha : g]) = ([\alpha + k_g],g)$, then ϕ is well defined. Because, $[\alpha : g] = [\alpha' : g']$ implies $g = g'$,α is homotopic to α' and hence $\alpha + k_g$ is homotopic to $\alpha' + k_g$.

Suppose $\phi([\alpha : g]) = \phi([\alpha' : g])$. Then $\alpha + k_g$ is homotopic to $\alpha' + k_g$. This implies that α is homotopic to α'. Therefore ϕ is injective.

For any element $([\alpha],g) \in \pi_1(X,f(x_0)) \times G$, there exists an element $[\alpha + k_g\rho : g]$ in $\sigma(X,f(x_0),G)$ such that $\phi([\alpha + k_g\rho : g]) = ([\alpha],g)$. Therefore, ϕ is surjective.

Next, we show that ϕ is a homomorphism. Let $[\alpha_1 : g_1]$ and $[\alpha_2 : g_2]$ be elements of $\sigma(X,f(x_0),G)$. Then

$$\phi([\alpha_1 : g_1] \circ [\alpha_2 : g_2]) = ([\alpha_1 + g_1\alpha_2 + k_{g_1g_2}],g_1g_2)$$

and

$$\phi([\alpha_1 : g_1]) \circ \phi([\alpha_2 : g_2]) = ([\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2}],g_1g_2).$$

Since $\alpha_2 + k_{g_2}$ is a loop at $f(x_0)$ and $k_{g_1}\rho$ is the traces of an 1_X-cyclic homotopy of order g_1 at $f(x_0)$, $\alpha_2 + k_{g_2}$ is homotopic to $k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1}$ by Lemma 4. Therefore, we have

$$\begin{aligned}\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2} &\sim \alpha_1 + k_{g_1} + k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1}\\ &\sim \alpha_1 + g_1(\alpha_2 + k_{g_2}) + k_{g_1}\\ &\sim \alpha_1 + g_1\alpha_2 + g_1k_{g_2} + k_{g_1}\\ &\sim \alpha_1 + g_1\alpha_2 + k_{g_1g_2}.\end{aligned}$$

This implies that ϕ is a homomorphism. Finally, we show ϕ sends $J(f,x_0,G)$ onto $J(f,x_0) \times G$. Let $[\alpha : g]$ be an element of $J(f,x_0,G)$.

Then there exists an f-cyclic homotopy $H : X \times I \longrightarrow X$ of order g with trace α and an 1_X-cyclic homotopy $J : X \times I \longrightarrow X$ of order g with trace $k_g\rho$.

Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ J(f(x), 2(1-t)), & 1/2 \le t \le 1. \end{cases}$$

Then $F(x,0) = H(x,0) = f(x), F(x,1) = J(f(x),0) = f(x)$ and

$$F(x_0,t) = \begin{cases} H(x_0, t), & 0 \le t \le 1/2 \\ J(f(x_0), 2(1-t)), & 1/2 \le t \le 1 \end{cases}$$
$$= (\alpha + k_g)(t).$$

Thus F is an f-cyclic homotopy with trace $F(x_0, \cdot)$ homotopic to $\alpha + k_g$ and hence $[\alpha + k_g]$ belongs to $J(f, x_0)$.

For any element $([\alpha], g) \in J(f, x_0) \times G$, there exists an f-cyclic homotopy $H : X \times I \longrightarrow X$ with trace α. Since $\{k_g | g \in G\}$ is a family of preferred 1_X-traces at $f(x_0)$, there exists an 1_X-cyclic homotopy $J : X \times I \longrightarrow X$ of order g with trace $k_g\rho$. Define

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ J(f(x), 2t-1), & 1/2 \le t \le 1. \end{cases}$$

then F is an f-cyclic homotopy of order g with trace $\alpha + k_g\rho$ and hence there exists an element $[\alpha + k_g\rho : g]$ in $J(f, x_0, G)$ such that $\phi([\alpha + k_g\rho : g]) = ([\alpha], g)$.

Let $[\alpha : g]$ be any element of $Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$. Thus $[\alpha : g]$ belongs to $J(f, x_0, G)$ if and only if $[\alpha + k_g]$ belongs to $J(f, x_0)$. Thus it is sufficient to show that $[\alpha + k_g] \in J(f, x_0)$. For any element $[\beta] \in \pi_1(X, x_0)$, we have $[f\beta + k_{g'}\rho : g'] = f_\sigma([\beta + h_{g'}\rho : g'])$ and hence $[f\beta + k_{g'}\rho : g']$ belongs to $f_\sigma(\sigma(X, x_0, G))$. Since $[\alpha : g]$ belongs to $Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$ we have

$$[\alpha : g] \circ [f\beta + k_{g'}\rho : g'] = [f\beta + k_{g'}\rho : g'] \circ [\alpha : g].$$

This implies

$$\begin{aligned} ([\alpha + k_g], g) \circ ([f\beta], g') &= \phi([\alpha : g]) \circ \phi([f\beta + k_{g'}\rho : g']) \\ &= \phi([f\beta + k_{g'}\rho : g'] \circ \phi([\alpha : g]) \\ &= ([f\beta], g') \circ ([\alpha + k_g], g). \end{aligned}$$

Therefore, we have $[\alpha+k_g]\circ[f\beta] = [f\beta]\circ[\alpha+k_g]$ for any $[\beta] \in \pi_1(X, x_0)$. Hence $[\alpha + k_g]$ belongs to $Z(f_\pi(\pi_1(X, x_0)), \pi_1(X, f(x_0))) = J(f, x_0)$ and this completes one part of the proof. The revrese implication follows from Theorem 5.

COROLLARY 9. *Let X be a connected aspherical polyhedron, G be abelian and f be an automorphism of (X, G). If (X, G) admits a family $\{k_g | g \in G\}$ of preferred 1_X-traces at $f(x_0)$, then $J(f, x_0, G) = Z(\sigma(X, f(x_0), G))$.*

COROLLARY 10. *Let $G = \{1_X\}$, X a connected aspherical polyhedron and f be a self-map of X, then $J(f, x_0) = Z(f_\pi \pi_1(X, x_0), \pi_1(X, f(x_0)))$.*

References

1. B. J. Jiang, *Lectures on Nielsen fixed point theory*, Contemp. Math. **14** (1983).
2. F. Rhodes, *On the fundamental group of a transformation group*, Proc. London Math. Soc. (3) **16** (1966), 635–650.
3. M. H. Woo, *A representation of $E(X, x_0, G)$ in terms of $G(X, x_0)$*, J. of Korean Math. Soc. **23** (1986), 101–108.
4. M. H. Woo and S. H. Han, *An extended Jiang subgroup of the fundamental group of a transformation group*, Comm. of Korean Math. Soc. **28** (1991), 135-143.

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