J. Korean Math. Soc. **31** (1994), No. 4, pp. 569–608

MEROMORPHIC FUNCTIONS, DIVISORS, AND PROJECTIVE CURVES: AN INTRODUCTORY SURVEY

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1. Introduction: an overview

The subject matter of this survey has to do with holomorphic maps from a compact Riemann surface to projective space, which are also called algebraic curves; the theory we survey lies at the crossroads of function theory, projective geometry, and commutative algebra (although we should mention that the present survey de-emphasizes the algebraic aspect). Algebraic curves have been vigorously and continuously investigated since the time of Riemann. The reasons for the preoccupation with algebraic curves amongst mathematicians perhaps have to do with other than the usual reason, namely, the herd mentality prompting us to follow the leads of a few great pioneering mathematicians in the field-the fact that algebraic curves possess a certain simple unity together with a rich and complex structure. From a differential-topological standpoint algebraic curves are quite simple as they are neatly parameterized by a single discrete invariant, the genus. Even the possible complex structures of a fixed genus curve afford a fairly complete description. Yet there are a multitude of diverse perspectives (algebraic, function theoretic, and geometric) often coalescing to yield a spectacular result.

The author's personal journey into algebraic curves began a few years ago when he was working on a problem in minimal surface theory [Y] : it turned out that the solution to Bob Osserman's problem of immersing a given punctured compact Riemann surface into $\mathbb{R}^3$ as a complete minimal surface depended on our ability to manufacture a meromorphic function with a prescribed polar divisor. It would not be an exaggeration to say that algebraic curves are responsible for many

Received July 1, 1993.

an inspiration in other branches of mathematics. And the author's main motivation for writing this survey, other than self-education, is to share the excitement and beauty of algebraic curves with an audience larger than that consisting of the experts.

We begin with a fixed compact Riemann surface $M = M_g$ of genus g. A meromorphic function on M is, by definition, a holomorphic map

$$f : M \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

Tradition dictates that $f(M) \neq \{\infty\}$. The set of all meromorphic functions on M will be denoted by $\mathrm{Hol}\,(M, \mathbb{P}^1)$. Identifying $\mathbb{C}$ with the constant functions on M we include $\mathbb{C}$ in $\mathrm{Hol}\,(M, \mathbb{P}^1)$. A fancier way to write the set of meromorphic functions on M is to write

$$H^0(M, \mathcal{M}^*) = \mathrm{Hol}\,(M, \mathbb{P}^1) \backslash \{0\},$$

where $\mathcal{M}^* \to M$ denote the sheaf of germs of not identically zero meromorphic functions on M.

Why study meromorphic functions? On a compact Riemann surface M the maximum principle for holomorphic functions prohibits any non-constant holomorphic function. We are thus led to study meromorphic functions on M, and more generally, holomorphic maps $M \to \mathbb{P}^r$. The totality of projective realizations of a fixed compact Riemann surface M turns out to have a variety structure; a careful study of this variety structure not only makes the nature of M transparent but also enriches our understanding of the set of all compact Riemann surfaces sharing the same genus.

A foundational result as regards the general structure of $\mathrm{Hol}\,(M, \mathbb{P}^r$ is the **Douady-Kuranishi Theorem.** For a complex manifold W, the set $S(W)$ of all compact complex submanifolds of W can be made into a complex space, called the Douady space. (This space is universal in a suitable sense.)

Roughly speaking, a complex space is a Hausdorff topological space that locally looks like an analytic subset of a complex domain; in particular, a nonsingular complex space is a complex manifold. Now let N_1 and N_2 be compact complex manifolds and put

$$\mathrm{Hol}\,(N_1, N_2) = \{\text{holomorphic maps } N_1 \to N_2\}.$$

By identifying $f \in \mathrm{Hol}(N_1, N_2)$ with its graph in $N_1 \times N_2$ we regard $\mathrm{Hol}(N_1, N_2)$ as an open subspace of $S(N_1 \times N_2)$, hence, making $\mathrm{Hol}(N_1, N_2)$ into a complex space. The Douady topology on $\mathrm{Hol}(N_1, N_2)$ turns out to coincide with the compact-open topology.

A holomorphic map $M \to \mathbb{P}^r$ is said to be degenerate if the image lies in a lower dimensional projective subspace. We exclude the degenerate curves from our consideration as otherwise they would appear in a redundant manner, thereby complicating the exposition.

Given a projective curve $M \to \mathbb{P}^r$ its degree is simply the number of intersection between the curve and a generic hyperplane $\mathbb{P}^{r-1} \subset \mathbb{P}^r$. In particular, the degree of a meromorphic function is the number of times the function takes on a generic value. We, therefore, have a natural stratification

$$\mathrm{Hol}(M, \mathbb{P}^r) = \bigcup_{d \geq 1} \mathrm{Hol}_d(M, \mathbb{P}^r),$$

where $\mathrm{Hol}_d(M, \mathbb{P}^r)$ consists of nondegenerate degree d curves. It is not difficult to show that each $\mathrm{Hol}_d(M, \mathbb{P}^r)$ is open and closed in $\mathrm{Hol}(M, \mathbb{P}^r)$.

We have thus narrowed our task to understanding the complex space $\mathrm{Hol}_d(M, \mathbb{P}^r)$. Our next trimming process is to divide $\mathrm{Hol}_d(M, \mathbb{P}^r)$ by the holomorphic automorphism group $\mathrm{Aut}(\mathbb{P}^r) \cong PGL(r+1)$. After all, two curves in $\mathbb{P}^r$ related by a projective automorphism can be brought together merely by a change of coordinates. The next logical step would be to mod out the automorphism group of M. But we do not do so for the following reason: when $\mathrm{Aut}(M)$ is not compact, for example, $\mathrm{Aut}(\mathbb{P}^1)$, the quotient $\mathrm{Hol}_d(M, \mathbb{P}^r)/(\mathrm{Aut}(\mathbb{P}^r) \times \mathrm{Aut}(M))$ is not Hausdorff. Moreover, when the genus $g \geq 2$, by a theorem of Schwarz the automorphism group $\mathrm{Aut}(M)$ is finite; hence, we are not gaining a great deal by the additional quotient process. We can now give the first formulation of our problem.

Problem 1. Given integers $d \geq 1$, $r \geq 1$, and a compact Riemann surface M of genus g, give an explicit description of the space

$$\mathrm{Hol}_d(M, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r).$$

An overriding observation in our investigation is that a projective curve is determined by a *linear series of divisors* up to a projective transformation. In a simpler vein, a meromorphic function is determined by its zero and polar divisors up to scalar multiplication. This observation, which is embodied in the fundamental correspondence theorem stated below, becomes all the more remarkable once it is seen what a simple object a divisor on a compact Riemann surface is.

DEFINITION. A divisor D on a compact Riemann surface M is a finite integral sum of points $p_i \in M$. We write

$$D = \Sigma a_i p_i, \quad a_i \in \mathbb{Z}.$$

Divisors occur naturally as the zeros and poles counted with multiplicity of meromorphic functions. Given a meromorphic function f on M, its divisor, denoted by (f), is defined to be its zero divisor (i.e., the zeros, each counted with multiplicity) minus its polar divisor. Suppose we have two meromorphic functions f and $\tilde{f}$ sharing the same divisor. Then the quotient $\tilde{f}/f$ is a nowhere vanishing holomorphic function, hence a constant. On the other hand, as we shall see in section 2 a divisor is the zero or the polar divisor of a meromorphic function if and only if it is integral and the complete linear series it defines has no base points. Thus the totality of meromorphic functions affords an elegant description in terms of divisors.

Two divisors D and $\tilde{D}$ are said to be linearly equivalent to each other if their difference is a principal divisor, meaning the divisor of a meromorphic function. Given a adivisor D the set of all integral divisors (i.e., the coefficients $a_i \geq 0$) linearly equivalent to it, called the complete linear series of D and denoted by $|D|$, is naturally a projective space. A projective subspace of a $|D|$ is called a linear series on M. The degree of a linear series is, by definition, the degree of a divisor in it, where the degree of a divisor is simply the sum of its coefficients. Note that the degree of a principal divisor must be zero by the equidistribution theorem for meromorphic functions, hence the degree of a linear series is well-defined. The set of all linear series on M with a fixed dimension r and degree d is denoted by $G^r_d = G^r_d(M)$, and an element in it is denoted by g^r_d. An important observation to make is that a linear series g^r_d on M defines a holomorphic map

$$\Phi : M \backslash B(g^r_d) \to (g^r_d)^* \cong \mathbb{P}^r, \quad p \mapsto H_p,$$

where $G(g_d^r)$ is the set of points common to all divisors, called the base locus of g_d^r, in g_d^r. Given any point $p \in M\backslash B(g_d^r)$ the set of all divisors in g_d^r containing p forms a hyperplane H_p in the projective space $g_d^r = \mathbb{P}^r$.

The fundamental correspondence theorem. Let g_d^r be a base-point-free linear series on M. Then the map $\Phi : M \to \mathbb{P}^r$, defined up to the choice of an identification $(g_d^r)^* = \mathbb{P}^r$, is a nondegenerate degree d projective curve. Conversely, let $f : M \to \mathbb{P}^r$ be a nondegenerate degree d curve. Then the set of all hyperplane sections on M, i.e., the divisors coming from intersecting $f(M)$ with hyperplanes in $\mathbb{P}^r$, is a linear series of dimension r degree d without a base point. Moreover, any two such projective curves related by an automorphism of $\mathbb{P}^r$ define the same linear series.

Let $F_d^r(M) \subset G_d^r(M)$ consists of linear series with base points. It turns out that this set is negligibly small. For example, $G_d^r(\mathbb{P}^1)$ is a complex space of dimension $(r+1)(d-r)$ and $F_d^r(\mathbb{P}^1)$ is a codimension r closed complex subspace. (In fact, $G_d^r(\mathbb{P}^1)$ is naturally identified with the Grassmann manifold of projective r-planes in $\mathbb{P}^d$, and $F_d^r(\mathbb{P}^1)$ is a codimension r subvariety.) At any rate, the dimension of $G_d^r(M)$ is unaffected by cutting out the closed complex subspace $F_d^r(M)$. We have arrived at the second formulation of our problem.

Problem 2. Give an explicit description of the set $G_d^r(M)\backslash F_d^r(M)$.

The following pages contain solutions, albeit decidedly partial, to Problems 1 and 2. In section 2 we lay a foundation for what is to follow: the Riemann-Roch theorem gives a relationship between the degree and the dimension of a linear series, which is precise when the degree is large compared with the genus; Abel's theorem yields a well-behaved fibration from the space of divisors (or that of line bundles) to the Jacobian variety, allowing us to take advantage of the abelian group structure of the Jacobian variety. Section 3 contains a fairly complete solution to Problems 1 and 2 for the case $r = 1$. In section 3 we give a two-step description of the space of degree d meromorphic functions on a fixed compact Riemann surface: Firstly, we map $\mathrm{Hol}_d(M, \mathbb{P}^1)$ to the Jacobian variety by taking the polar divisor; we then describe the image and the fiber of this map. The last section contains an exposition, mostly without proofs, of the famous Brill-Noether theorem, due to

Brill-Noether, Severi, Kempf, Kleiman-Laksov, and Griffiths-Harris. To be more precise, we discuss the

Dimension Theorem. For a general Riemann surface M of genus g, the complex space $\mathrm{Hol}_d(M, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$ has dimension $\rho(g, r, d)$, where

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is the Brill-Noether number.

In the above theorem it is agreed that when ρ is negative the set $\mathrm{Hol}_d(M, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$ is empty-this, incidentally, is the part proved by Griffiths and Harris. The existence implication for $\rho \geq 0$, in fact, holds for every Riemann surface of genus g with the understanding that the dimension could be larger than ρ for certain surfaces. All we do in section 4 is to sketch a proof of this existence implication.

We now talk a little about what is not covered in this survey. In this survey we are interested in holomorphic maps from a fixed compact Riemann surface M to $\mathbb{P}^r$ of a fixed degree d. If one varies the Riemann surface M, then the resulting object is, more or less. what is known as the Hilbert scheme, denoted by $H_{d,g,r}$. Thus our survey has to do with describing the general fiber of the map

$$H_{d,g,r} \rightarrow \mathcal{M}_g,$$

where $\mathcal{M}_g$ is the moduli of all compact Riemann surfaces of genus g. By now there are several different ways to view $\mathcal{M}_g$ (Teichmüller, Schottky, and Deligne-Mumford), and we do not entertain the moduli question in this survey (see [H] for a masterful, albeit advanced, account of the moduli problem). Another interesting and important subject we do not delve into is that of non-general curves, e.g., hyperelliptic and trigonal curves (see [K] and references cited therein). We will be quite content with grappling with a description of $\mathrm{Hol}_d(M, \mathbb{P}^r)$ for a general Riemann surface M.

Nearly everything in this survey is a well-known result, at least to an expert. The book [ACGH] gives an authoritative account the Brill-Noether theory; the book [N] gives an expert account of the deformation techniques for studying the Douady-Kuranishi space. Yet an introductory survey such as this, the author felt, would serve the important purpose of introducing this exceedingly beautiful subject matter to a larger audience-after all, we have tried to keep the prerequisites at a minimum while maintaining a consistent persepective.

2. Foundational material: the Riemann-Roch Theorem and Abel's Theorem

Holomorphic line bundles on a compact Riemann surface provide, at the least, a convenient and unifying language in the study of divisors. So we begin this section with a discussion of the relationship between line bundles and divisors, which then leads to a discussion of the Riemann-Roch theorem.

Let $\mathrm{Pic}(M)$ denote the set of all line bundles on M, which forms a group under the tensor product. $\mathrm{Pic}(M)$ is called the Picard group of M. Note that the inverse of a line bundle L is given by its dual L^* since $L \otimes L^*$ is the trivial bundle. Another description of $\mathrm{Pic}(M)$ is $\mathrm{Pic}(M)$ as the Cech cohomology group $H^1(M, \mathcal{O}^*)$, where $\mathcal{O}^*$ denotes the sheaf of germs of nowhere vanishing holomorphic functions on M: To see this one needs first to observe that transition functions $\{g_{ab}\}$ of a line bundle $L \to M$ relative to an open cover $\mathcal{U} = \{U_a\}$ of M define a Cech 1-cocyle $\in Z^1(N(\mathcal{U}), \mathcal{O}^*)$, where $N(\mathcal{U})$ is the nerve of $\mathcal{U}$; then apply a direct limit argument. With the identification $\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)$ in mind we will often write

$$L \otimes L' = L + L', \quad L^* = -L.$$

To make the relationship between line bundles and divisors clear it is helpful to view divisors in sheaf-theoretic terms as well. For this we take a divisor D and write

$$D = \Sigma a_i p_i - \Sigma b_j q_j = D^+ - D^-,$$

where the p_i's and q_j's are distinct points and $a_i > 0$, $b_j > 0$. Given an open cover $\mathcal{U} = \{U_a\}$ of M the divisor D is given by local data $\{f_a \in \mathcal{M}^*(U_a)\}$, where $\mathcal{M}^*(U_a)$ is the set of meromorphic functions on U_a: the zeros of f_a are given by D^+ restricted to U_a and the poles of f_a are given by D^- restricted to U_a. Clearly, on an overlap $U_a \cap U_b$

the quotient f_a/f_b is a nonvanishing holomorphic function. We have, therefore, arrived at the identification

$$\mathrm{Div}(M) = H^0(M, \mathcal{M}^*/\mathcal{O}^*),$$

where $\mathrm{Div}(M)$ denotes the group of all divisors on M. We are now able to define the homomorphism

$$\mathrm{Div}(M) \to H^1(M, \mathcal{O}^*), \quad D = \{f_a\} \mapsto L_D = \{g_{ab} = f_a/f_b\}.$$

It is easy to see that the kernel of this homomorphism consists of principal divisors, and we obtain a monomorphism

$$\mathrm{Div}(M)/\sim \hookrightarrow H^1(M, \mathcal{O}^*), \quad [D] \mapsto L_D.$$

We shall see shortly that this map is, in fact, an epimorphism, allowing us the identification

$$\mathrm{Div}(M)/\sim = H^1(M, \mathcal{O}^*), \quad [D] = L_D.$$

The holomorphic sections of a line bundle $L \to M$ is denoted by $H^0(L)$, and we will let $h^0(L)$ denote its dimension. The holomorphic cotangent bundle, also called the canonical bundle, will be denoted by $K \to M$. Given a line bundle L its degree can be defined as the integral

$$\int_M c_1(L), \quad c_1(L) = \text{ the Chern class of } L.$$

Another method, which is more useful in our setting, of finding the degree of a line bundle is to take the degree of a divisor representing it. The famous Riemann-Roch theorem then gives a relationship between the dimension of $H^0(L)$ and the degree of L.

The Riemann-Roch theorem for line bundles. Let L be a line bundle on a genus g Riemann surface M. Then

$$h^0(L) - 1 = \deg L - g + h^0(K \otimes L^*).$$

Using the Riemann-Roch theorem we can now see that every line bundle $L \to M$ comes from a divisor: For any point $p \in M$ we can find an integer n such that the line bundle $L \otimes$ [np] ([np] denotes the line bundle assoicated with the divisor np) admits a nontrivial holomorphic section-take n to be large and use the Riemann-Roch theorem. This shows that L has a nontrivial meromorphic section ξ; hence, L is associated to a divisor, namely, the divisor of ξ.

Given a divisor $D = D^+ - D^- \in \mathrm{Div}(M)$ there arises the complex vector space

$$L(D) = \{f \in H^0(M, \mathcal{M}^*) : (f) \geq -D\} \cup \{0\}.$$

A nonzero element of $L(D)$ is a meromorphic function whose zero divisor is at least D^- and whose polar divisor is at most D^+. Fix a meromorphic section s_0 of the line bundle $L_D = [D] \to M$. This gives rise to an isomorphism

$$H^0([D]) \to L(D), \quad \eta \mapsto \eta/s_0.$$

(A minor but technically important observation here is that the quotient of any two meromorphic sections of a line bundle is a well-defined meromorphic function on M. We can also make the identification somewhat more natural by choosing s_0 so that its divisor is D.) For example, when D is integral (meaning that $D^- = 0$) we may think of $H^0([D])$ as meromorphic functions on M whose polar divisors are at most D.

The complete linear series of D is, by definition,

$$|D| = \{D' \geq 0 : D' \sim D\}.$$

The sets $|D|$ and the projective space $\mathbb{P}(L(D))$ are naturally identified via

$$D' \in |D| \mapsto \text{ the line through } f \text{ in } L(D),$$

where f is the meromorphic function satisfying $D + (f) = D'$. We thus obtain an identification $|D| = \mathbb{P}(L(D))$.

The sets $|D|$ and $\mathbb{P}(H^0[D])$ can be directly identified as follows: A divisor D' in $|D|$ corresponds to the line of holomorphic sections

$$\{\lambda \cdot \eta : 0 \neq \lambda \in \mathbb{C}\},$$

where the zero divisor of $\eta \in H^0([D])$ is D'.

By way of notation we put

$$r(D) = \dim |D|,$$
$$i(D) = h^0(K \otimes [-D]).$$

The integer $i(D)$ is called the index of specialty of D and will play an important role in our later analysis. We can now restate the Riemann-Roch theorem in divisor terms.

The Riemann-Roch theorem for divisors. Let D be any divisor on a genus g Riemann surface M. Then

$$r(D) = \deg D - g + i(D).$$

Noting that $L(D) = 0$ for a negative degree divisor D we can easily calculate the dimension $r(D)$ in extreme degree ranges. If d denotes the degree of the divisor D, then

$$r(D) = \begin{cases} -1, & \text{if } d < 0, \\ d - g, & \text{if } d > 2g - 2 \end{cases}$$

with the usual convention that the empty set has a negative dimension.

We now want to talk about the fundamental correspondence theorem giving a description of holomorphic maps in terms of linear series. Let $L = [D]$ be a line bundle on M and consider a base-point-free linear series $g_d^r \subset |D|$. Its affinization is given by a $(r+1)$-dimensional vector subspace $\Lambda \subset H^0([D])$. Pick a basis $\eta_0, \cdots, \eta_r$ of Λ. Observing that a point of M lies in the common zero locus of the η_i's if and only if it is a base point of g_d^r we see that the holomorphic map

$$\Phi(g_d^r) : M \to \mathbb{P}^r, \quad p \mapsto [\eta_0(p), \cdots, \eta_r(p)]$$

is well-defined. Another choice of a basis for Λ would have the effect of replacing Φ by $A \circ \Phi$, where A is an automorphism of $\mathbb{P}^r$. Since the η_i's form a basis it is clear that the image $\Phi(g_d^r)(M) \subset \mathbb{P}^r$ is nondegenerate. It is also easy to see that the curve $\Phi(g_d^r)(M)$ intersects a generic hyperplane $\mathbb{P}^{r-1} \subset \mathbb{P}^r$ exactly d times.

Conversely, consider a nondegenerate degree d curve $f : M \to \mathbb{P}^r$. A hyperplane section on M is, by definition, the pullback of a hyperplane divisor on $\mathbb{P}^r$. To put it another way, a hyperplane section D on M is the intersection of $f(M)$ with a hyperplane in $\mathbb{P}^r$ counted with multiplicity. Thus the holomorphic curve f gives rise to a linear series, namely

$$\{D \in \operatorname{Div}(M) : D \text{ is a hyperplane section}\}.$$

By construction, this linear series is a g_d^r; moreover, $\Phi(g_d^r) = f$ with a suitable choice of coordinates, namely the original coordinates used in giving f.

By way of notation we set

$$G_d^r(M) = \{g_d^r\text{'s on } M\}, \quad F_d^r(M) = \{g_d^r\text{'s with base points }\},$$
$$\operatorname{Hol}_d(M, \mathbb{P}^r) = \{\text{nondegenerate curves in } \mathbb{P}^r \text{ of degree } d\}.$$

The fundamental correspondence theorem then states that

$$G_d^r(M)\backslash F_d^r(M) \cong \operatorname{Hol}_d(M, \mathbb{P}^r)/\operatorname{Aut}(\mathbb{P}^r).$$

Let Z be a canonical divisor on M. A canonical divisor is, by definition, the divisor of a meromorphic 1-form on M. In particular, if $f \in H^0(M, \mathcal{M}^*)$, then its differential df is a meromorphic 1-form and its divisor (df) is a canonical divisor. Considering the Laurent expansions of f near its poles and ramification points and then applying the Riemann-Hurwitz formula one sees easily that the degree of (df) is $2g - 2$. In fact, any two canonical divisors are linearly equivalent to each other, hence the degree of any canonical divisor is $2g - 2$. The canonical linear series on M, usually denoted by $|K|$, consists of all integral divisors linearly equivalent to a canonical divisor. Thus

$$|K| = \mathbb{P}(H^0(K)),$$

where K is the canonical bundle on M. A theorem of Hodge states that

$$H^0(K) \cong \mathbb{C}^g,$$

where g is the genus of M.

The canonical map. Let us assume that $g \geq 2$ so that the canonical series $|K|$ of M is at least a pencil, meaning that its dimension is at least one. Given any linear series $|D|$ there is a simple criterion to see if a point $p \in M$ is a base point. One always has

$$r(D) - r(D-p) = 0 \text{ or } 1,$$

and this number is zero exactly when p is a base point of $|D|$. Now the Riemann-Roch theorem tells us that

$$h^0(K \otimes [-p]) = g - 1,$$

and from this we can conclude that the canonical series is base-point-free. The canonical map is then the holomorphic map given by

$$\Phi(|K|) : M \to \mathbb{P}^{g-1},$$

which is determined up to Aut $(\mathbb{P}^{g-1})$. Suppose now that M does not carry a g_2^1, i.e., $G_2^1(M) = \emptyset$. Then the Riemann-Roch theorem can be used to show that the canonical map is an embedding, giving a god-given projective realization of the Riemann surface M. The Riemann surface M is said to be hyperelliptic if $G_2^1(M) \neq \emptyset$. In such a case it is well-known that the canonical map is a ramified two-sheeted cover of the rational normal curve $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$. At any rate, the hyperelliptic surfaces are quite special and far better understood than the general Riemann surfaces. We will see later that the only g_{2g-2}^{g-1} on M is the canonical series, that is to say, $G_{2g-2}^{g-1}(M)$ is the singleton $\{|K|\}$.

Suppose we have a divisor $D = \Sigma p_i$, where the p_i's are all distinct. Given an embedding $\psi : M \to \mathbb{P}^r$ we set

$$S_{\psi,D} = \text{the span of the points } \psi(p_i) \subset \mathbb{P}^r.$$

Recalling that $i(D)$ is nothing but the number of independent holomorphic differentials ω with $(\omega) \geq D$ we see that

$$i(D) = g - 1 - \dim S_{\varphi,D},$$

where $\varphi : M \to \mathbb{P}^{g-1}$ denotes the canonical map. Combining this with the Riemann-Roch theorem we obtain

The Riemann-Roch theorem for the canonical map. Let $D = \Sigma p_i$ be a divisor on a Riemann surface of genus $g \geq 2$. Then

$$r(D) = d - 1 - \dim S_{\varphi, D}.$$

Rephrasing, $r(D)$ is the number of independent linear relations amongst the points $\varphi(p_i)$ on the canonical curve.

REMARK. The above formulation of the Riemann-Roch theorem is also known as the geometric version [ACGH, p. 12].

The Riemann-Roch theorem for the canonical map allows us to determine $r(D)$ in the degree range $0 \leq g \leq 2g - 2$, assuming that D is a general integral divisor.

$$\dim S_{\varphi,D} = \begin{cases} d-1, & \text{if } 0 < d \leq g, \\ g-1, & \text{if } g \leq d \leq 2g-2. \end{cases}$$

It follows that for a general integral D,

$$r(D) = \begin{cases} 0, & \text{if } 0 < d \leq g, \\ d-g, & \text{if } g \leq d \leq 2g-2. \end{cases}$$

To put it another way, the "exceptional" divisors lie in the degree range $0 < d \leq 2g - 2$; they are exactly those divisors whose points counted with multiplicity fail to be independent on the canonical image. It is worth noting that a canonical divisor is exceptional in this sense since a general integral divisor of degree $2g-2$ would have $r(D) = g-2$. We will have a lot more to say about exceptional divisors in later sections.

The other key ingredient-one key ingredient being the Riemann-Roch theorem-in the theory of divisors is Abel's theorem and the accompanying theory of the Jacobian variety. The underlying philosophy here is to map everything to the Jacobian variety $J(M)$ and exploit the fact that $J(M)$ is an abelian complex Lie group. By now a classic reference on the Jacobian variety exemplifying this philosophy is [*Gu*]. Our discussion will focus upon the object $W_d^r \subset J(M)$ which parametrizes the set of all complete linear series of degree d and dimension at least r. Regrettably, we will have to leave unexplored many exciting aspects of the theory, including the theta divisor and Riemann's theorem, and the Schottky problem.

Given $\alpha, \beta \in H_1(M, \mathbb{Z})$ we let $\sharp(\alpha, \beta)$ denote their intersection number. In terms of their Poincaré duals $\xi_\alpha, \xi_\beta \in H^1_{dR}(M)$ the intersection number is given by the integral

$$\sharp(\alpha, \beta) = \int_M \xi_\alpha \wedge \xi_\beta \in \mathbb{Z}.$$

A basis (e_g), $1 \leq a \leq 2g$, of $H_1(M, \mathbb{Z})$ is called a canonical homology basis, or a symplectic basis, if the intersection matrix $(\sharp(e_a, e_b))$ is given by

$$J_g = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

By Poincaré duality and using the fact that each de Rham cohomology class contains a unique harmonic representative we can find a basis (ζ^i), $1 \leq i \leq g$, of $H^0(K)$ such that the $g \times 2g$ matrix $(P^i_a) = (\int_{e_a} \zeta^i)$ is given by

$$(P^i_a) = (I_g, II),$$

where II is symmetric with a positive definite imaginary part. This fact, in turn, can be used to show that the Jacobian variety of M (defined below) satisfies so called the Riemann conditions, showing that it is a projective manifold.

A vector in $\mathbb{C}^g$ of the form

$$\int_\alpha (\zeta^i), \quad \alpha \in H_1(M, \mathbb{Z}),$$

is called a period of M with respect to the basis (ζ^i). So the set of all periods of M is exactly the lattice L generated by the $2g$ columns of $P = (I_g, II)$, and the Jacobian variety of M is defined to be

$$J(M) = \mathbb{C}^g / L.$$

Given a canonical homology basis $(e_a)_{1 \leq a \leq 2g}$ the condition

$$\int_{e_j} \zeta^i = \delta^i_j, \quad 1 \leq i,\ j \leq g,$$

uniquely specifies the basis (ζ^i) of $H^0(K)$ due to Poincaré duality. From this we see that the Jacobian variety $J(M)$ depends only on the choice of a canonical homology basis. On the other hand, from the intersection matrix J_g we see that any two canonical homology bases $e, \tilde{e}$ are related by

$$\tilde{e} = e \cdot X, \quad X \in Sp(g, \mathbb{Z}),$$

where $Sp(g,\mathbb{Z})$ is the subgroup of $SL(2g,\mathbb{Z})$ given by

$$Sp(g,\mathbb{Z}) = \{X \in GL(2g,\mathbb{Z}) : {}^tX \cdot J_g \cdot X = J_g\}.$$

The Siegel upper half space of genus g is defined to be $\mathbb{H}_g = \{Z \in GL(g,\mathbb{C}) : Z$ is symmetric with a positive definite imaginary part $\}$ so that the last $2g$ columns II of a period matrix (I_g, II) define a point of $\mathbb{H}_g$. The action of $Sp(g,\mathbb{Z})$ on the set of canonical homology bases of M induces in a natural way an action on $\mathbb{H}_g$. To be precise, if we write

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(g,\mathbb{Z}),$$

then

$$X(Z) = (C + DZ) \cdot (A + BZ)^{-1}, \quad Z \in \mathbb{H}_g.$$

From this we surmise that there is a well-defined map

$$\begin{aligned} j : \mathcal{M}_g = \{\text{compact Riemann surfaces of genus } g\} \to \mathbb{H}_g/Sp(g,\mathbb{Z}), \\ M \mapsto j(M) = II \text{ (modulo the action of } Sp(g,\mathbb{Z})). \end{aligned}$$

We can now state the famous

Torelli theorem. Compact Riemann surfaces M and M' of genus g are conformally equivalent to each other if and only if $j(M) = j(M')$.

What is disheartening in Torelli's theorem is the fact that the map j is not surjective when the genus g is larger than 3. The problem of explicitly identifying the image of the map j is generally refered to as the Schottky problem (see [AD] or [S] for a recent discussion of this problem.)

We are now in a position to give the Jacobi map

$$\varphi : M \to J(M).$$

We will need to fix a base point $p_0 \in M$. Then for any $p \in M$ we consider the vector of integrals given by

$$^t\Big(\int_{p_0}^{p} \zeta^1, \cdots, \int_{p_0}^{p} \zeta^g\Big) \in \mathbb{C}^g.$$

The ambiguity involved in writing the line integral $\int_{p_0}^{p} \zeta^i$ disappears once we pass to the quotient space, $\mathbb{C}^g$ modulo the period lattice. Thus

$$\varphi(p) = {}^t\Big(\int_{p_0}^{p} \zeta^1, \cdots, \int_{p_0}^{p} \zeta^g\Big) \text{ (modulo the periods) } \in J(M).$$

Extending this definition linearly we define

$$\varphi : \mathrm{Div}(M) \to J(M).$$

It is interesting to note that once restricted to the subgroup $\mathrm{Div}^0(M) \subset \mathrm{Div}(M)$ consisting of degree zero divisors the Jacobi map no longer depends on the choice of a base point p_0.

Let $z^1, \cdots, z^g$ be the Euclidean coordinates on $\mathbb{C}^g$ and note that the dz^i's are well-defined holomorphic differentials on the complex torus $\mathbb{C}^g/L = J(M)$. Then since the complete linear series $|D|$ is a projective space, the pullbacks $\varphi^* dz^i$ must vanish on $|D|$, showing that the Jacobi map $\mathrm{Div}^d_+(M) \to J(M)$ must be constant on the complete linear series $|D|$. Abel's theorem says more.

Abel's theorem. Let $D_1, D_2 \in \mathrm{Div}^d_+(M)$. Then D_1 is linearly equivalent to D_2 if and only if $\varphi(D_1) = \varphi(D_2)$.

Abel's theorem tells us exactly that the fibers of the Jacobi map

$$\varphi : \mathrm{Div}^d_+(M) \to M$$

are complete linear series. Let $\mathrm{Pic}^d(M) \subset H^1(M, \mathcal{O}^*)$ denote the subset of the Picard group consisting of degree d line bundles. Note that $\mathrm{Pic}^d(M)$ is identified with $\mathrm{Div}^d_+(M)$ modulo linear equivalence via

$$L_D = [D] \in \mathrm{Pic}^d(M) \mapsto |D| \in \mathrm{Div}^d_+(M)/\sim.$$

Abel's theorem amounts to the statement that the induced map

$$\mathrm{Pic}^d(M) \to J(M), \quad [D] \mapsto \varphi(D)$$

is well-defined and injective. As for the surjectivity we have the

Jacobi inversion theorem. The Jacobi map $\varphi : \mathrm{Div}^d_+(M) \to J(M)$ is onto when the degree d is at least the genus g.

Noting that $J(\mathbb{P}^1)$ is a point, we suppose the genus to be positive. Consider the Jacobi map

$$\varphi : M \to J(M) = \mathbb{C}^g/L.$$

We will show that this map is immersive by showing that its lifting to $\mathbb{C}^g$, denoted by $\tilde{\varphi}$, is. Let $p \in M$ be arbitrary and also let z be a local coordinate centered at p. We then have

$$\tilde{\varphi}(z) = {}^t(\psi^1(z), \cdots, \psi^g(z)) \in \mathbb{C}^g.$$

where

$$\psi^i(z) = \int_{p_0}^{p} \zeta^i + \int_0^z \eta^i(z)dz, \quad \zeta^i = \eta^i dz.$$

So $d\psi^i/dz = \eta^i$ and ψ is non-immersive at z if and only if the η^i's all vanish at z, i.e., the ζ^i's all vanish at z. But then z would be a base point of the canonical linear series, which is not possible. This shows that $\varphi|_M$ is a holomorphic immersion. Now suppose for some $p, q \in M$ we had $\varphi(p) = \varphi(q)$. Then $\varphi(p - q) = 0$ and by Abel's theorem $p - q$ would be a principal divisor, meaning that there is a degree one meromorphic function on M which would force $M = \mathbb{P}^1$. It follows that φ embeds M into $J(M)$.

We now consider the Jacobi map defined on the degree d integral divisors

$$\varphi : \mathrm{Div}^d_+(M) \to J(M)$$

Take a generic point $D = p_1 + \cdots + p_d \in \mathrm{Div}^d_+(M)$ with the p_i's all distinct. Then for local coordinates z_i centered at p_i on M the collection $(z_1, \cdots, z_d)$ coordinatizes $\mathrm{Div}^d_+(M)$. So for $E = \Sigma z_i$ near D we have

$$\frac{\partial_\varphi}{\partial_{z_i}}(E) = \frac{\partial_{\tilde{\varphi}}}{\partial_{z_i}}(E) = \frac{\partial}{\partial_{z_i}}\left(\Sigma_k \int^{z_k} \zeta^1, \cdots, \Sigma_k \int^{z_k} \zeta^g\right)$$
$$= \left(\frac{\zeta^1}{dz_i}, \cdots, \frac{\zeta^g}{dz_i}\right)(E) = \left(\frac{\zeta^j}{dz_i}\right)(E).$$

Thus the Jacobian matrix of the map φ at the point E is given by

$$\left(\frac{\zeta^j}{dz_i}\right)(E), \quad 1 \le i \le d, \quad 1 \le j \le g.$$

Changing the local coordinate z_i has the effect of changing the i-th column of the above Jacobian matrix by a nonzero factor. Pick p_1 such that $\zeta^1(p_1) \neq 0$ and subtract a suitable multiple of ζ^1 from (ζ^i) making $\zeta^i(p_i) = 0$ for $i > 1$. Proceeding this way we can make the Jacobian matrix triangular. The following result, which is essentially the Jacobi inversion theorem stated above, now follows easily:

$$\dim W_d(M) = d, \text{ if } d \le g;$$
$$W_d(M) = J(M), \text{ if } d \ge g,$$

where $W_d = \varphi(\mathrm{Div}^d_+(M)) \subset J(M)$. Note that Abel's theorem tells us that $W_d \cong \mathrm{Pic}^d(M)$ parametrizes the set of all complete linear series of degree d. By the proper mapping theorem, W_d, being the holomorphic image of the compact complex manifold $\mathrm{Div}^d_+(M)$, is an analytic irreducible subvariety of $J(M)$. Moreover,

$$W_d = W_1 + \cdots + W_1 \ (d \text{ times})$$

since $\varphi(p_1 + \cdots + p_d) = \varphi(p_1) + \cdots + \varphi(p_d)$.

By way of notation we put

$$C^r_d = \{D \in \mathrm{Div}^d_+(M) : r(D) \ge r\},$$
$$W^r_d = \varphi(C^r_d) \subset W_d, \quad d \ge 1, \ r \ge 0.$$

Thus an important consequence of our discussion in this section is that the Jacobi map induces a fiberation

$$C^r_d \backslash C^{r+1}_d \to W^r_d \backslash W^{r+1}_d$$

with standard fiber $\mathbb{P}^r$.

3. Meromorphic functions on a compact Riemann surface

Fix a compact Riemann surface M of genus g. For a positive integer d we let

$$R_d(M) = \mathrm{Hol}_d(M, \mathbb{P}^1) \subset \mathrm{Hol}(M, \mathbb{P}^1)$$

denote the set of degree d meromorphic functions on M. Observe that $R_1(M)$ is empty unless $M = \mathbb{P}^1$ since any degree one meromorphic function would give a homeomorphism $M \to \mathbb{P}^1$. On the other hand, considering a non-Weierstrass point we see that $R_d(M)$ is nonempty whenever $d > g$. Since this observation will resurface in a more general setting we will not go into a discussion of Weierstrass points. Suffice it to say that all but finitely many points on a Riemann surface are non-Weierstrass points, and at a non-Weierstrass point p there is a meromorphic function with polar divisor $d \cdot p$ for any given $d > g$. The interested reader may consult [FK, pp. 76-86] for a detailed discussion of this.

Before embarking on the structural study of $R_d(M)$ we would like to mention a motivational theme for the present discussion, namely the Mittag-Leffler problem: The problem is to find a meromorphic function whose polar divisor is specified beforehand. This problem arises in many different contexts in application. A favorite example of the author [Y] comes from the theory of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. The famous Chern-Osserman theorem states that if $S \subset \mathbb{R}^3$ is a complete minimal surface, then S is of finite total curvature if and only if it is *algebraic*, meaning that S is a finitely punctured compact Riemann surface and the Gauss map $S \to \mathbb{P}^1$ is holomorphically extended to the compact Riemann surface M containing S. A crucial observation is that the Gauss map has poles on the puncture set $\Sigma = M \backslash S$; consequently, the problem of minimally (and conformally) immersing a compact Riemann surface M with punctures at a specified set Σ depends strongly on our ability to come up with meromorphic functions with poles at Σ.

The basic fact regarding the Mittag-Leffler problem is that the higher the degree of the given divisor the easier it is to find the meromorphic functions, which is related to the previously established fact that when the degree is large there are no special divisors. In fact, we will see a little later that when the degree is larger than $2g$, any integral divisor is the polar divisor of a meromorphic function.

Let us begin our study of $R_d(M)$ with a simple but instructive example.

EXAMPLE. Consider a meromorphic function f of degree $d > 0$ on the extended plane $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. The function f is, by definition, a holomorphic map

$$f : \mathbb{C} \cup \{\infty\} \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

In terms of the inhomogeneous coordinate $z = z_1/z_0$ on the domain and the inhomogeneous coordinate on the range $w = w_1/w_0$ we can write the map f as

$$z \mapsto w = Q(z)/P(z), \quad \infty \mapsto \lim_{z \to \infty} Q(z)/P(z),$$

where the polynomials P and Q have no common factors and

$$\max(\deg P, \deg Q) = d.$$

Homogenization of z and w leads to

$$f : [z_0, z_1] \mapsto [f_0(z_0, z_1), f_1(z_0, z_1)],$$

where

$$\begin{aligned} f_0 &= a_0 z_0^d + a_1 z_0^{d-1} z_1 + \cdots + a_d z_1^d, \text{ and} \\ f_1 &= b_0 z_0^d + b_1 z_0^{d-1} z_1 + \cdots + b_d z_1^d \end{aligned}$$

are such that the resultant polynomial $R(a_0, \cdots, a_d, b_0, \cdots, b_d) \neq 0$. If follows that there is a (biholomorphic) identification

$$\begin{aligned} R_d(\mathbb{P}^1) &\to \mathbb{P}^{2d+1} \backslash \Delta, \\ f &\mapsto [a_0, \cdots, a_d, b_0, \cdots, b_d] \in \mathbb{P}^{2d+1}, \end{aligned}$$

where Δ is the hypersurface in $\mathbb{P}^{2d+1}$ given by the resultant polynomial.

We now proceed to show that for any compact Riemann surface M of genus g the dimension of $R_d(M)$ is $2d - g + 1$, provided that $d \geq g$. We begin with the elementary but important observation that a meromorphic function on M is determined, up to multiplication by a nonzero scalar, by its divisor

$$(f) = (f)_0 - (f)_\infty.$$

To see this one merely notes that the ratio of any two meromorphic functions with the same divisor must be a nowhere zero holomorphic function on M, hence a constant. Observe, moreover, that the divisors $(f)_0$ and $(f)_\infty$ have no points in common. Conversely, if D_1 and D_2 are any two integral divisors that are linearly equivalent to each other with no points in common, then there is a meromorphic function f with $(f) = D_1 - D_2$: this is a restatement of Abel's theorem. Thus it is natural to view $R_d(M)/\mathbb{C}^*$ in divisor theoretic terms.

Recall that $\mathrm{Div}_+^d(M)$, being biholomorphic to the d-fold symmetric product of M, is a d-dimensional compact complex manifold. Consider the map

$$\Phi : \mathrm{Div}_+^d(M) \times \mathrm{Div}_+^d(M) \to J(M), \quad (D_1, D_2) \mapsto \varphi(D_1) - \varphi(D_2).$$

So the image of Φ is given by $W_d - W_d \subset J(M)$. FOr $D \in \mathrm{Div}_+^d(M)$ put

$$Y_D = \{D\} \times \varphi^{-1}(x), \quad x = \varphi(D) \in J(M).$$

If $d \geq g$, then by Jacobi Inversion $\varphi^{-1}(x)$ is $(d - g)$-dimensional. Now

$$\Phi^{-1}(0) = \bigcup_D Y_D,$$

and, more importantly,

$$R_d(M)/\mathbb{C}^* = \Phi^{-1}(0) \setminus \Sigma,$$

where Σ is the irreducible hypersurface (try to prove this) of $\mathrm{Div}_+^d(M)$ $\times \mathrm{Div}_+^d(M)$ given by

$$\Sigma = \{(D_1, D_2) : \ \mathrm{supp}(D_1) \cap \ \mathrm{supp}(D_2) \neq \emptyset\}.$$

But since the dimension of Y_D is $d - g$ it follows that

$$\dim R_d(M) = 2d - g + 1.$$

A little more work yields the following

THEOREM. *For $d \geq g$, $R_d(M)$ is a complex manifold of dimension $2d - g + 1$.*

The following result is thematic.

PROPOSITION. *Let D be an integral divisor. Then the complete linear series $|D|$ has no base points if and only if D is the polar divisor of a meromorphic function.*

Proof. If $D = (f)_\infty$ for a meromorphic function f, then $(f)_0 \in |D|$ and D have no points in common. Conversely, assume that $|D|$ is base-point-free. We first consider the case where $|D|$ is a pencil. Then D can be realized as the polar divisor of the meromorphic function $\Phi_{|D|} : M \to \mathbb{P}^1$ by suitably choosing a basis of $H^0([D])$, where $\Phi_{|D|}$ is the holomorphic curve associated to $|D|$. Suppose now that the dimension of $|D|$ is larger than one, and consider the nondegenerate holomorphic map

$$\Phi_{|D|} : M \to \mathbb{P}^N, \quad N = \dim |D|.$$

Every element of $|D|$ is a hyperplane section. In particular, D comes from a hyperplane $H \subset \mathbb{P}^N$. Choose another hyperplane H' such that H' does not intersect $H \cap \Phi_{|D|}(M)$. Consider the pencil of divisors on M given by

$$\{D_\lambda = \lambda_0 H + \lambda_1 H' : \lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}.$$

Here, $\lambda_0 H + \lambda_1 H'$ stands for the hyperplane section coming from the hyperplane

$$\lambda_0 F(z_0, \cdots, z_N) + \lambda_1 G(z_0, \cdots, z_N) = 0,$$

where the linear forms F and G define H and H' respectively. We then note that for any point $p \in M$ there is a unique $\lambda(p) \in \mathbb{P}^1$ such that $p \in \text{support}(D_\lambda)$, meaning that the assignment $f : p \mapsto \lambda(p)$ is a holomorphic map. We can now take $D = (f)_\infty$. $\square$

It is convenient to introduce the notation

$$\mathrm{Bpf}^d_+(M) = \{D \in \mathrm{Div}^d_+(M) : |D| \text{ is base-point-free}\} \subset C^1_d.$$

PROPOSITION. *Let D be any integral divisor of degree $d \geq 2g$. Then there is a meromorphic function whose polar divisor (or zero divisor) is D, that is to say,*

$$Bpf^d_+(M) = Div^d_+(M), \quad d \geq 2g.$$

Proof. Let $D \in \mathrm{Div}^d_+(M)$ with $d \geq 2g$. Then by Riemann-Roch, $r(D) = d - g$. On the other hand, we know that for any point $p \in M$,

$$r(D) - r(D-p) = 0 \text{ or } 1;$$

moreover, $r(D) - r(D-p) = 0$ if and only if p is a base point of $|D|$. But, again by Riemann-Roch

$$r(D-p) = (d-1) - g$$

and the result follows. $\square$

When $d < 2g$ the situation is more complicated-a generic divisor of degree less than $2g$ is not the polar divisor of a meromorphic function.

Another interesting situation is that of $\mathrm{Div}^{2g-2}_+(M)$. Here we put

$$\mathrm{Can}^{2g-2}_+(M) = \{D \in \mathrm{Div}^{2g-2}_+(M) : D \text{ is a canonical divisor}\}.$$

Given $D \in \mathrm{Div}^{2g-2}_+(M)$ we let D' denote a divisor *residual* to D, meaning that

$$D' = Z - D$$

for some canonical divisor Z. Then the degree of D' is zero, and according to whether or not D' is principal we have the dimension of $r(D) = 0$ or -1. Since D' is principal if and only if D is canonical we see that

$$r(D) = g - 1, \text{ if } D \text{ is canonical;}$$
$$r(D) = g - 2, \text{ if } D \text{ is not canonical.}$$

Thus $D \in \mathrm{Can}^{2g-2}_+(M)$ if and only if $|D|$ is a g^{g-1}_{2g-2}. We have thus shown that G^{g-1}_{2g-2} consists of the singleton $|K|$. Since the canonical linear series is base-point-free

$$\mathrm{Can}^{2g-2}_+(M) \subset \mathrm{Bpf}^{2g-2}_+(M).$$

Now consider the Jacobi map

$$\varphi : \mathrm{Div}_{+}^{2g-2}(M) \to J(M).$$

Letting $\kappa \in J(M)$ denote the point corresponding to the canonical series we have

$$\varphi^{-1}(\kappa) = \mathrm{Can}_{+}^{2g-2}(M), \quad \dim \mathrm{Can}_{+}^{2g-2}(M) = g - 1.$$

Note, in particular, that a generic integral divisor of degree $2g - 2$ is not canonical.

We now turn to the space $R_d(M)/\mathrm{Aut}\,(\mathbb{P}^1)$, which turns out to be more appropriate than $R_d(M)/\mathbb{C}^*$ in its capacity for generalization to a holomorphic curve setting. Consider the map

$$\alpha : R_d(M) \to \mathrm{Div}_{+}^{d}(M) \to J(M), \quad f \mapsto D = (f)_\infty \mapsto \varphi(D).$$

An easily verified fact is that for any $f \in R_d(M)$ and $A \in \mathrm{Aut}\,(\mathbb{P}^1)$ the divisors $(f)_\infty$ and $(a \circ f)_\infty$ are linearly equivalent (see below for a proof). From this we surmise that the map α descends to a map

$$\tilde{\alpha} : R_d(M)/\mathrm{Aut}\,(\mathbb{P}^1) \to J(M).$$

Note also that $x \in J(M)$ is in the image of α if and only if the complete linear series $\varphi^{-1}(x)$ is base-point-free by an earlier proposition. And we are interested in examining the set

$$\alpha(R_d(M)) \cap (W_d^r \backslash W_d^{r+1}) \subset J(M), \quad r \geq 1.$$

This set parameterized the set of complete linear series of degree d and dimension exactly r that are base-point-free. That is to say,

$$\alpha(R_d(M)) \cap (W_d^r \backslash W_d^{r+1}) = \{\text{complete } g_d^r\text{'s without base points}\}.$$

The *gap subvariety* is defined by

$$F_d^r = W_{d-1}^r + W_1 \subset J(M),$$

where "+" denotes the group sum in $J(M)$. Since the W_i's are additive we see that $F_d^r \subset W_d^r$. For any $x \in W_d^{r+1}$ and $p \in M$ we put

$$\Lambda_{x,p} = \{E' \in \mathrm{Div}_+^{d-1}(M) : E' + p \in \varphi^{-1}(x)\}.$$

Note that the linear series

$$\Lambda_{x,p} + p = \{E \in \varphi^{-1}(x) : p \in \mathrm{supp}(E')\},$$

that is to say, $\Lambda_{x,p} + p$ consists of all divisors in $\varphi^{-1}(x)$ supporting p. Now the dimension of $\varphi^{-1}(x)$ is at least $r + 1$. So, the dimension of $\Lambda_{x,p}$ is at least r as $\Lambda_{x,p} + p$ is at least a hyperplane in $\varphi^{-1}(x)$. This shows that

$$x = \varphi(\Lambda_{x,p}) + \varphi(p) \in W_{d-1}^r + W_1 = F_d^r.$$

Since x was an arbitrary element of W_d^{r+1} we have shown that $W_d^{r+1} \subset F_d^r$. So

$$W_d^{r+1} \subset F_d^r \subset W_d^r.$$

We can now give a satisfactory description of the set of base-point-free complete g_d^r's, at least at the level of the Jacobian variety.

THEOREM. *The set of base-point-free g_d^r's is parameterized, via the Jacobi map, by $W_d^r \backslash F_d^r \subset J(M)$. In other words,*

$$\alpha(R_d(M)) \cap (W_d^r \backslash W_d^{r+1}) = W_d^r \backslash F_d^r.$$

Proof. We know that $W_d^r \backslash F_d^r \subset W_d^r \backslash W_d^{r+1}$ and it is enough to show that

$$F_d^r \backslash W_d^{r+1} = \{x \in W_d^r \backslash W_d^{r+1} : \varphi^{-1}(x) \text{ is a } g_d^r \text{ with a base point}\}.$$

Certainly if $x \in W_d^r \backslash W_d^{r+1}$ is such that $\varphi^{-1}(x)$ has a base point p, then we can write $x = y + y_1$ for some $y \in W_{d-1}^r$ and $y_1 = \varphi(p) \in W_1$. To prove the other containment take an arbitrary $x \in F_d^r \backslash W_d^{r+1}$. By definition, we can find $y \in W_{d-1}^r$ and $y_1 \in W_1$ such that $x = y + y_1$. Put $p = \varphi^{-1}(y_1) \in M$. Then

$$\varphi^{-1}(x) = \varphi^{-1}(y) + p = \{E' + p : E' \in \mathrm{Div}_+^{d-1}(M) \cap \varphi^{-1}(y)\}.$$

Since $\varphi^{-1}(x)$ has dimension exactly r, the dimension of $\varphi^{-1}(y)$, too, is r. But this means that p is a base point of $\varphi^{-1}(x)$. $\square$

Consequently, we obtain a natural stratification of the image of α, namely,

$$\alpha(R_d(M)) = \bigcup_{r\geq 1} W_d^r \backslash F_d^r.$$

Our next task then is to understand the totality of meromorphic function lying over a single point $x \in J(M)$. We begin this with an elementary

LEMMA. *Let $f \in R_d(M)$, and $A \in Aut(\mathbb{P}^1)$. Then the divisors $(f)_\infty$ and $(A \circ f)_\infty$ are linearly equivalent.*

Proof. Think of $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$. Then $A \circ f$ can be written as

$$z \in M \mapsto w = f(z) \in \mathbb{C} \mapsto \frac{aw+b}{cw+d} \in \mathbb{C},$$

and in case $w = \infty, a(w) = \lim_{w\to\infty} \frac{aw+b}{cw+d}$, where $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} \neq 0$. From this the result follows rather easily. For example, taking the generic case $ac \neq 0$ we have

$$(f)_\infty = (T_{-a/c} \circ A \circ f)_0,$$

where $T_{-a/c} : w \mapsto w - a/c$ But

$$(T_{-a/c} \circ A \circ f)_0 \sim (T_{-a/c} \circ A \circ f)_\infty \sim (A \circ f)_\infty,$$

where $\sim$ denotes the linear equivalence. $\square$

Let $D = (f)_\infty$, where $f \in R_d(M)$, and consider the complete linear series

$$|D| \cong \mathbb{P}^r, \quad r = r(D).$$

We will let L denote the projective line in $|D|$ through the points $(f)_\infty$ and $(f)_0$. Let A be a fixed but otherwise arbitrary automorphism of $\mathbb{P}^1$, and put

$$\tilde{f} = A \circ f \in R_d(M).$$

Then the above lemma tells us that the projective line $\tilde{L}$ passing through $(\tilde{f})_0$ and $(\tilde{f})_\infty$ is also contained in $|D|$. We claim that $L = \tilde{L}$. To begin with L corresponds to a 2-plane in $L(D)$,

$$L = \mathbb{P}\left(\operatorname{span}\{1, f\}\right).$$

Likewise,

$$\tilde{L} = \mathbb{P}\left(\operatorname{span}\{1, \tilde{f}\}\right).$$

But

$$\begin{aligned}\mathbb{P}\left(\operatorname{span}\{1, \tilde{f}\}\right) &= \mathbb{P}\Big(\operatorname{span}\Big\{1, \frac{af+b}{cf+d}\Big\}\Big)\\ &= \mathbb{P}\left(\operatorname{span}\{cf+d, af+b\}\right) = \mathbb{P}\left(\operatorname{span}\{1, f\}\right).\end{aligned}$$

The next observation is that given any two points $D_1 \neq D_2 \in L$ there is an automorphism $A \in \operatorname{Aut}(\mathbb{P}^1)$ such that $D_1 = (A \circ f)_0$ and $D_2 = (A \circ f)_\infty$. To see this let $D_i, 1 \leq i \leq 2$, correspond to the 2-plane

$$\operatorname{span}\{a_i f + b_i\} \subset L(D).$$

Then we can take

$$A(w) = (a_1 w + b_1)/(a_2 w + b_2).$$

COROLLARY. *Let $x \in W_d^1$ and consider a projective line L in $\varphi^{-1}(x)$. Then*

$$\text{either } L \times L \subset \Sigma, \quad \text{or } (L \times L) \cap \Sigma = \Delta_{L \times L},$$

where Δ is the diagonal set and $\Sigma \subset Div_+^d(M) \times Div_+^d(M)$ is the "exceptional" set introduced earlier.

Proof. Suppose $(D_1, D_2) \in (L \times L) \backslash \Sigma$. Then $(D_1, D_2) \in \Phi^{-1}(0)$, meaning that $D_1 - D_2$ is a principal divisor. $\square$

We note that generically we will have $(L \times L) \cap \Sigma = \Delta_{L \times L}$.

OBSERVATION. L is base-point-free if and only if $(L \times L) \cap \Sigma = \Delta$.

Proof. Suppose $(L \times L) \cap \Sigma = \Delta$ and take $(D_1, D_2) \in (L \times L) \backslash \Delta$. Then $(D_1, D_2) \in \Phi^{-1}(0)$. So there is a meromorphic function f with $(f)_0 = D_1$ and $(f)_\infty = D_2$. Thus $\mathrm{supp}(D_1) \cap \mathrm{supp}(D_2) = \emptyset$, showing that L could not possibly have a base point. Conversely, assume that L is base-point-free. Suppose we had $(D_1, D_2) \in (L \times L) \cap \Sigma$ with $D_1 \neq D_2$. Then there is a point $p \in M$ common to both D_1 and D_2. But then p would be in the base locus of the linear pencil

$$\tilde{L} = \mathbb{P}\{aD_1 + bD_2 : a, b \in \mathbb{C}\}.$$

But we have seen earlier that this pencil is exactly L. $\square$

Let $x \in W_d^1$, and also let $G(1, \varphi^{-1}(x))$ denote the Grassmann manifold of projective lines in $\varphi^{-1}(x)$. We have a Zariski open subset of $G(1, \varphi^{-1}(x))$ given by

$$Z(1, \varphi^{-1}(x)) = \Big\{L \in G(1, \varphi^{-1}(x)) : (L \times L) \cap \Sigma = \Delta\Big\}.$$

Moreover, this set consists exactly of base-point-free pencils in $\varphi^{-1}(x)$. We can now given the main result of this section.

THEOREM. *Assume that $\varphi^{-1}(x)$ has no base points. Then the Zariski open set $Z(1, \varphi^{-1}(x)) \subset G(1, \varphi^{-1}(x))$ parametrizes the projective equivalence classes of meromorphic functions lying over $x \in J(M)$. To put it another way, there is an isomorphism*

$$\Psi : \tilde{\alpha}^{-1}(x) \subset R_d(M)/Aut(\mathbb{P}^1) \to Z(1, \varphi^{-1}(x)).$$

Proof. Let $f \in R_d(M)$ be so chosen that $(f)_\infty \in \varphi^{-1}(x)$. To f we associate the line $L_f \in Z(1, \varphi^{-1}(x))$ through $(f)_0$ and $(f)_\infty$. Then the assignment

$$f \in \alpha^{-1}(x) \mapsto L_f$$

projects down to give a map.

$$\Psi : f \text{ (modulo Aut}(\mathbb{P}^1)) \in \tilde{\alpha}^{-1}(x) \mapsto L_f \in Z(1, \varphi^{-1}(x)).$$

This map is seen to be well-defined from the earlier observation that $L = \tilde{L}$. To see that the map is surjective, take a pencil $L \subset \varphi^{-1}(x)$ that is base-point-free. Then for any $(D_1, D_2) \in (L \times L)\backslash\Delta$ we know that $D_1 - D_2$ is a principal divisor, say (f). Thus L comes from f via Ψ. Now suppose $L_f = L_{f'}$. To show that our assingment is injective we need to show that $f \equiv f'$ (modulo $\mathrm{Aut}(\mathbb{P}^1)$). But any two point on the line $L_f = L_{f'}$ come from $A \circ f$ for some A, showing that $f' = A \circ f$ for some $A \in \mathrm{Aut}(\mathbb{P}^1)$. $\square$

Assume that the degree is large, say $d \geq 2g$, and consider the projections

$$\tilde{\alpha} : R_d(M)/\mathrm{Aut}(\mathbb{P}^1) \to W_d^{d-g} = J(M),$$
$$\varphi : \mathrm{Div}_+^d(M) \to J(M).$$

The Jacobi map, then, is a fibration with standard fiber $\mathbb{P}^{d-g}$. Moreover, the preceding theorem tells us that the map $\tilde{\alpha}$, too, is nearly a fibration: the fiber over a point $x \in W_d^1$ is a generic subset of the Grassmannian $G(1, \mathbb{P}^{d-g})$.

We now proceed to give another, somewhat more useful, description of the fiber $\tilde{\alpha}^{-1}(x)$, which may be considered as a dual description to the one given in the above theorem. But we need first to review the notion of a projection centered at a subspace, which occurs frequently in Algebraic Geometry. Let II_1 be an n-plane in $\mathbb{P}^N$, and also let II_2 be any $(N - n - 1)$-plane not intersecting II_1. Then the projection centered at the subspace II_1 is the holomorphic map given by

$$\pi : \mathbb{P}^N \backslash II_1 \to II_2, \quad p \mapsto II_{p,II_1} \cap II_2,$$

where II_{p,II_1} denotes the $(n+1)$-plane through p and II_1. Since the set of all $(n+1)$-planes in $\mathbb{P}^N$ containing II_1 is naturally a $\mathbb{P}^{N-n-1}$ we can also define the projection simply as

$$\pi : \mathbb{P}^N \backslash II_1 \to \mathbb{P}^{N-n-1}, \quad p \mapsto II_{p,II_1}.$$

Coming back to $x \in W_d^1 \subset J(M)$, put

$$r = \dim \varphi^{-1}(x) \geq 1.$$

If $r = 1$, then $\tilde{\alpha}^{-1}(x)$ is either empty or a singleton for dimensional reasons. So let us assume that $r > 1$, i.e., $x \in W^2_d$. Assuming that x lies in the image of $\tilde{\alpha}$-meaning that $\varphi^{-1}(x)$ has no base points-the associated holomorphic curve $\Phi_x(M)$ is a nondegenerate degree d curve in $\mathbb{P}^r$. It is natural to consider $\varphi^{-1}(x)$ as the dual space $\mathbb{P}^{r*}$ to this $\mathbb{P}^r$ as every divisor in $\varphi^{-1}(x)$ is the pullback by Φ_x of a hyperplane in $\mathbb{P}^r$. A line L in $\varphi^{-1}(x)$ then cooresponds to a $(r-2)$-subspace S_L of $\mathbb{P}^r$. Moreover, we have

LEMMA. *The condition* $(L \times L) \cap \Sigma = \Delta$ *dualizes to the condition*

$$\Phi_X(M) \cap S_L = \emptyset.$$

Proof. Recall the canonical identification

$$\mathbb{P}(\mathrm{Hom}(\mathbb{C}^{r+1}, \mathbb{C})) = \mathbb{P}^{r*}, \quad [\theta] \mapsto \mathrm{Ker}(\theta).$$

So if $L_{\alpha,\beta}$ is a line in $\mathbb{P}^{r*}$ through the hyperplanes α, β in $\mathbb{P}^r$, then

$$S_L = \alpha \cap \beta \subset \mathbb{P}^r.$$

So the condition that $L = L_{\alpha,\beta}$ be base-point-free means that the hyperplane sections α and β have no points in common, meaning that $\alpha \cap \beta$ does not intersect the curve $\Phi_x(M)$. $\square$

Consequently, the projection centered at $S = S_L$ defines a meromorphic function

$$\pi_S : \Phi_x(M) \to \mathbb{P}^1.$$

We define a Zariski open set Z of $G(r-2, \mathbb{P}^r)$ by

$$Z^*_x = \{S \in G(r-2, \mathbb{P}^r) : \Phi_x(M) \cap S = \emptyset\}$$

so that $Z^*_x \cong Z(1, \varphi^{-1}(x) = \mathbb{P}^{r*})$. We can now restate the above theorem as

THEOREM. *Suppose* $x \in W^2_d$ *with* $\varphi^{-1}(x)$ *base-point-free. Then*

$$\tilde{\alpha}^{-1}(x) \cong Z^*_x, \quad f \ (\mathit{mod}\ \mathit{Aut}(\mathbb{P}^1)) \mapsto S_f = S_{L_f}.$$

Moreover, f is obtained as the composition

$$\pi_{S_f} \circ \Phi_x : M \to \mathbb{P}^r \to \mathbb{P}^1.$$

Again for $d \geq 2g$ we are saying that the fiber at a point x of the projection

$$\tilde{\alpha} : R_d(M)/\mathrm{Aut}\,(\mathbb{P}^1) \to J(M)$$

is a generic subset Z_x^* of the Grassmannian $G(r-2, \mathbb{P}^r = \varphi^{-1}(x)^*)$. In addtion, this time, we are giving a rather explicit prescription for writing down the meromorphic functions in $\alpha^{-1}(x)$ in terms of Φ_x and certain projections $\mathbb{P}^r \to \mathbb{P}^1$.

Take $M = \mathbb{P}^1$ so that $J(M)$ is a point. Put $[0,1] = \infty \in \mathbb{P}^1$, and consider the divisor $D = d \cdot \infty \in \mathrm{Div}_+^d(\mathbb{P}^1)$. We then have

$$L(D) = \mathrm{span}\,\{1, z, \cdots, z^d\},$$

where by z^i we mean the meromorphic function $[1,z] \mapsto [1,z^i]$. The holomorphic curve $\Phi_{|D|} : \mathbb{P}^1 \to \mathbb{P}^d$ is thus given by

$$[1,z] \mapsto [1, z, z^2, \cdots, z^d], \quad \infty \mapsto [0, \cdots, 0, 1].$$

The image $C = \Phi_{|D|}(\mathbb{P}^1)$ is called the rational normal curve. Then for $d \geq 2$,

$$R_d(\mathbb{P}^1)/\mathrm{Aut}\,(\mathbb{P}^1) \cong Z^* = \{S \in G(d-2, \mathbb{P}^d) : S \cap C = \emptyset\}.$$

In particular, $R_2(\mathbb{P}^1)/\mathrm{Aut}\,(\mathbb{P}^1)$ is a $\mathbb{P}^2$ minus a conic. Recalling that $R_2(\mathbb{P}^1)$ is a $\mathbb{P}^5$ minus the resultant hypersurface we have the following intriguing

COROLLARY. *There is a $PGL(2)$-principal fibration*

$$\mathbb{P}^5 \backslash V \to \mathbb{P}^2 \backslash C,$$

where V is a quartic hypersurface and C is a conic curve.

4. The Brill-Noether number and the dimension theorem

We saw already the identification $\mathrm{Hol}_d(M, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r) = G_d^r \backslash F_d^r$, where G_d^r consists of linear series, complete or not, of degree d and dimension exactly r, and the subvariety F_d^r consists of such linear series with base points. On the other hand, we saw that $\varphi(C_d^r) = W_d^r \subset J(M)$ parametrizes the space of complete linear series of degree d and dimension at least r. And we have a fair understanding of the structure of W_d^r with the aid of such theorems as the Riemann-Roch theorem and Abel's theorem, assuming that d is large compared with the genus. In this section we will talk about W_d^r in the lower degree range, discussing Clifford's theorem and the Brill-Noether number along the way. But before doing this we would like to mention one important fact we will use without proof. That is, the fact that G_d^r is obtained from W_d^r via a blow-up process (in fact, G_d^r is called the canonical blow-up of W_d^r). This blow-up construction is somewhat involved in terms of the machinery required, and is discussed in detail in [ACGH, chapters 2 and 4]. Suffice it to say that W_d^r and G_d^r are isomorphic complex spaces outside a small set, and their dimensions coincide.

Again we begin with a motivational example. Take $M = \mathbb{P}^1$. Then since the Jacobian variety is a singleton we see that any two integral divisors of degree d are linearly equivalent to each other, i.e.,

$$\mathrm{Div}_+^d(\mathbb{P}^1) = |d \cdot \infty|.$$

As before we will think of $|d \cdot \infty|$ as the dual space $\mathbb{P}^{d*}$ to the projective space $\mathbb{P}^d$ in which the associated holomorphic curve $\Phi_{|d\cdot\infty|}(\mathbb{P}^1)$ lies. Now

$$G_d^r = G(r, |d \cdot \infty|),$$

and a brute force calculation shows that

$$\mathrm{codim}\ F_d^r = r.$$

Thus we have

$$\mathrm{Hol}_d(\mathbb{P}^1, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r) \cong G(r, \mathbb{P}^d) \backslash W,$$

where W is a codimension r subvariety. This example exhibits the unplesant phenomenon of the moduli space not being compact. On the other hand, it also indicates that the moduli space is nearly compact so that it can be compactified in a nice way.

We can be more explicit in our description of $\mathrm{Hol}_d(\mathbb{P}^1, \mathbb{P}^r)$. Put

$$Z(r, \mathbb{P}^{d*} = |d\cdot\infty|) = \{L \in G(r, |d\cdot\infty|) : L \text{ is base-point-free}\} = G_d^r \backslash F_d^r.$$

This is a Zariski open subset of the Grassmannian G_d^r. (Recall that when $r = 1$, the base-point-free condition is exactly the condition $(L \times L) \cap \Sigma = \Delta$ so that our notations are consistent.) Given an r-plane $L \in Z(r, \mathbb{P}^{d*})$ let S_L denote the $(d-r-1)$-plane in $\mathbb{P}^d$ dual to it. Then the condition that L be base-point-free translates into the condition that $S = S_L$ not intersect the rational normal curve $C_d = \Phi_{|d\cdot\infty|}(\mathbb{P}^1) \subset \mathbb{P}^d$ so that we have a well-defined projection

$$\pi_L = \pi_S : C^d \subset \mathbb{P}^d \backslash S_L \to \mathbb{P}^r.$$

All this is summarized by the identification

$$\begin{gathered} Z(r, |d\cdot\infty|) \cong \mathrm{Hol}_d(\mathbb{P}^1, \mathbb{P}^r)/\mathrm{Aut}\,(\mathbb{P}^r), \\ L \mapsto \pi_L \circ \Phi_{|d\cdot\infty|}. \end{gathered}$$

THEOREM. *Any nondegenerate holomorphic curve $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d arises as the rational normal curve in $\mathbb{P}^d$ followed by the projection centered at a generic subspace of codimension $r+1$.*

Coming back to the general case let M be any Riemann surface of genus g, and recall

$$C_d^r = \{D \in \mathrm{Div}_+^d(M) : r(D) \geq r\}.$$

The basic observation is that the Jacobi map gives rise to a $\mathbb{P}^r$-fibration

$$C_d^r \backslash C_d^{r+1} \to W_d^r \backslash W_d^{r+1}.$$

Moreover, since C_d^{r+1} and W_d^{r+1} are subvarieties we have

$$r + \dim W_d^r = r + \dim G_d^r = \dim C_d^r.$$

Let us look at the easy case, namely, the case $d \geq 2g$. We then know that

$$r(D) = d - g, \quad D \in \mathrm{Div}^d_+(M);$$

in particular, $r(D)$ is a constant. Moreover,

$$C_d^{d-g+1} = \emptyset, \quad W_g^{d-g+1} = \emptyset$$

so that we obtain a $\mathbb{P}^{d-g}$-fibration

$$\varphi : C_d^{d-g} = \mathrm{Div}^d_+(M) \to W_d^{d-g} = J(M).$$

Thus

$$\dim C_d^r = d = g + (d - g), \quad \dim W_d^{d-g} = g. \tag{$\dagger$}$$

What we would like to do is to generalize the dimension calculation ($\dagger$) to the case $d \leq 2g - 1$. The first difficulty one runs into here is that $r(D)$ is no longer a constant. However, an upper bound on $r(D)$ is easily found.

Clifford's Theorem. Let $D \in \mathrm{Div}^d_+(M)$ with $d \leq 2g - 1$. Then

$$r(D) \leq d/2.$$

We will give a proof following [ACGH, p. 108]. If $i(D) = 0$, then the result follows easily from the Riemann-Roch theorem. So we assume that $i(D) > 0$. A useful observation to make is the following : $r(D) \geq n$ if and only if there is a divisor in $|D|$ containing any n given points. As a consequence, we have for any two effective divisors D and D',

$$r(D + D') \geq r(D) + r(D') :$$

given $r(D) + r(D')$ points of M we can find $E \in |D|$ containing the first $r(D)$ and $E' \in |D'|$ containing the remaining $r(D')$ points so that $E + E' \in |D + D'|$ contains the given set of points. Now since D is assumed to be special we may find an integral divisor D' such that $D + D'$ is a canonical divisor. Since $i(D) = r(D') + 1$, we then obtain

$$r(D) + r(D') \leq r(D + D') = g - 1,$$
$$r(D) - r(D') = d - g + 1,$$

and the Clifford inequality follows.

It turns out, however, that the Clifford upper bound is rather crude, at least for divisors on a general Riemann surface. Indeed if D is an integral divisor of degree $\leq 2g-1$ lying on a general Riemann surface of genus g, then

$$r(d) = r^2 + r(g+1-d) - d \leq 0.$$

Note that the graph of $r = r(d)$ in the rd-plane is a parabola lying strictly below the straight line $r = d/2$ in the degree range $0 \leq d \leq 2g-1$. The condition that $r(d) \leq 0$ is exactly the condition that $\rho(g, r, d) \geq 0$.

Let C_d^r be as in the above, where we no longer assume that $d \geq 2g$. We can now state the

Existence Theorem. Every component of $C_d^r \subset \mathrm{Div}_+^d(M)$ has dimension at least $\rho + r$, assuming that $\rho \geq 0$

We will sketch a proof of the above theorem following [ACGH, pp.159-160]. Let

$$D = \Sigma p_i \in \mathrm{Div}_+^d(M),$$

and suppose that z is a local coordinate on M defined in an open set containing the points $p_1, \cdots, p_d$. We can then write

$$\zeta_i(z) = \eta_i(z)dz, \quad 1 \leq i \leq g,$$

where the ζ^i's are a basis for $H^0(K_M)$. Consider the $g \times d$ matrix

$$\mathcal{J}(D) = \begin{bmatrix} \zeta_1(p_1), & \cdots, & \zeta_1(p_d) \\ & \vdots & \\ \zeta_g(p_1), & \cdots, & \zeta_g(p_d) \end{bmatrix} = \begin{bmatrix} \eta_1(z(p\ )), & \cdots, & \eta_1(z(p_d)) \\ & \vdots & \\ \eta_g(z(p\ )), & \cdots, & \eta_g(z(p_d)) \end{bmatrix}$$

Now suppose that the p_i's supporting D are all distinct, and put

$$\sharp = \text{the rank of the above matrix}.$$

Then since $i(D)$ is the number of independent holomorphic differentials vanishing at the points p_i we have

$$i(D) + \sharp = g.$$

On the other hand, by the Riemann-Roch theorem

$$i(D) = r(D) - g + d.$$

It follows that

$$r(D) \geq r \text{ if and only if } \sharp \leq d - r.$$

We thus have a local description of the subvareity $C_d^r \subset \mathrm{Div}_+^d(M)$ near a generic point. Namely, near a generic D the subvariety C_d^r is realized as the common zero locus of the $(d-r+1) \times (d-r+1)$ minors of the matrix $\mathcal{J}(D)$. This linear algebraic description of C_d^r will allows us to calculate a lower bound on the dimension.

Digression: Determinantal varieties. In what follows we will give a calculation showing that the dimension of the variety consisting of $m \times n$ matrices of rank at most k is $k(m+n-k)$. Along the way we will also indicate how G_d^r may be realized as a blow-up of W_d^r. To aid the reader, who may wish to consult [ACGH, chapters 2 and 4] to learn more about this material, we will stick to the notations in [ACGH]. Let $M(m,n)$ denote the projectivized space of $m \times n$ complex matrices, which can be identified with $\mathbb{P}^{mn-1}$. For $0 \leq k \leq \min(m,n)$ we denote by $M_k \subset M(m,n)$ the subvariety of matrices of rank at most k : M_k is called the k-th generic determinatal subvariety. Set

$$\tilde{M}_k = \{(A, W) \in M(m,n) \times G(n-k,n) : A(W) = 0\}.$$

It is not hard to see that the projection $\tilde{M}_k \to G(n-k,n)$ makes $\tilde{M}_k$ into a holomorphic vector bundle of rank mk over the Grassmannian $G(n-k,n)$. In particular, $\tilde{M}_k$ is a smooth connected complex manifold of dimension $k(m+n-k)$. Under the projection

$$\pi : M(m,n) \times G(n-k,n) \to M(m,n)$$

$\tilde{M}_k$ gets mapped properly onto M_k; by the proper mapping theorem and the connectedness of $\tilde{M}_k$ it then follows that M_k is an irreducible algebraic subvariety of $M(m,n)$. Now let $A \in M_k \backslash M_{k-1}$ be arbitrary. Then the fiber of π over A is the singleton $(A, \operatorname{Ker} A) \in \tilde{M}_k \subset M(m,n) \times G(n-k,n)$. Since $M_k \backslash M_{k-1}$ is a generic subset of M_k this shows that $\tilde{M}_k$ is a *desingularization* of M_k (in fact, it can be shown that the singular locus of M_k is exactly M_{k-1}). In particular,

$$\dim M_k = \dim \tilde{M}_k = k(m + n - k).$$

Let $E, F \to X$ be holomorphic vector bundles of rank n and m respectively, over the complex space X. Consider a holomorphic bundle map

$$\varphi : E \to F.$$

Locally the map φ is given by an $m \times n$ matrix A_φ of holomorphic functions. So φ amounts locally to a holomorphic map

$$A_\varphi : U \subset X \to M(m, n).$$

Put $U_k = A_\varphi^{-1}(M_k)$ and define a subvariety $X_k(\varphi) \subset X$ by the prescription

$$X_k(\varphi) \cap U = U_k.$$

Thus

$$X_k(\varphi) = \{p \in X : \text{ rank } (A_{\varphi,x}) \leq k\}.$$

$X_k(\varphi)$ is called the k-th determinantal variety associated with φ. By construction either $X_k(\varphi)$ is empty or its codimension is at most that of M_k, i.e.,

$$\operatorname{codim} X_k(\varphi) \leq (m - k)(n - k).$$

Put

$$\begin{aligned}\tilde{X}_k(\varphi) = \{(x, W) : x \in X_k(\varphi), W \in G(N - k, E_x), W \subset \text{ Ker } A_{\varphi,x}\} \\ \subset G(n - k, E),\end{aligned}$$

where $G(n - k, E)$ is the Grassmann bundle of $(n - k)$-planes in E. Under the projection

$$G(n - k, E) \to X$$

$\tilde{X}_k(\varphi)$ gets mapped to $X_k(\varphi)$. In fact, for an open set $U \subset X$

$$\tilde{X}_k(\varphi) \cap \pi^{-1}(U) = U \times \tilde{M}_k.$$

$\tilde{X}_k(\varphi)$ is called the canonical blow-up of $X_k(\varphi)$. Note that when the base space X is smooth, so is $\tilde{X}_k(\varphi)$

REMARK. The variety W_d^r can be made into a determinantal variety in $X = W_d \subset J(M)$ [ACGH, p. 177], and when this is done one can show that $\tilde{W}_d^r = G_d^r$.

Proof of the Existence Theorem. Since C_d^r is given by the simultaneous vanishing of every $(d-r+1)\times(d-r+1)$ minor of a $d \times g$ matrix of holomorphic functions its codimension may not exceed

$$(d-(d-r))\cdot(g-(d-r)) = r\cdot(g+d+r). \qquad \square$$

We now know that for any Riemann surface of genus g

$$\dim G_d^r \geq \rho(g,d,r).$$

In fact, the equality holds for a general Riemann surface. Moreover, Griffiths and Harris [GH] showed that for a general Riemann surface the set G_d^r is empty when $\rho(d,g,r) < 0$. In summary, we can thus say that for a general Riemann surface M of genus g, the set $\mathrm{Hol}_d(M,\mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$ is a projective manifold of dimension ρ (isomorphic to G_d^r minus a proper subvariety of relatively high codimension).

We conclude our survey with a brief mention of the smoothness question. Gieseker [*Gi*] showed that for a general Riemann surface G_d^r is smooth. Gieseker's theorem and other smoothness results rely on the notion of semi-regularity which we define below. A divisor $D \in C_d^r$ is said to be semi-regular if the cup product homomorphism

$$H^0([D]) \otimes H^0([K-D]) \to H^0(K), \quad \xi\otimes\eta \mapsto \xi\cdot\eta$$

is injective. Similarly a line bundle $L \in W_d^r$ is said to be semi-regular if the cup product homomorphism

$$H^0(L)\otimes H^0(K\otimes L^*) \to H^0(K)$$

is injective. Given a linear series $g_d^r \in G_d^r$ let $W \subset H^0([D])$ be the corresponding $(r+1)$-dimensional vector space, where $g_d^r \subset |D|$. Then the linear series g_d^r is said to be semi-regular if the restricted cup product homomorphism

$$W \otimes H^0([K-D]) \to H^0(K)$$

is injective. We can now state the

Semi-Regularity theorem.

(1) C_d^r is smooth at $D \in C_d^r \backslash C_d^{r+1}$ and has the expected dimension $r+\rho$ if and only if D is semi-regular. (When $g-r+d \geq 0$, no component of C_d^r lies entirely in C_d^{r+1} so that C_d^{r+1} is very small.)

(2) W_d^r is smooth of dimension ρ at $L \in W_d^r \backslash W_d^{r+1}$ if and only if L is semi-regular. (Again, when $g-r+d \geq 0$, no component of W_d^r lies in W_d^{r+1}.)

(3) G_d^r is smooth of dimension ρ at g_d^r if and only if g_d^r is semi-regular.

The overall idea behind the proof of the semi-regularity theorem is to give a description of the tangent space in terms of certain cohomology groups.

A Few Words on the Literature. The literature on algebraic curves is incredibly vast, and we do not pretend to be familiar with even a small portion of it. In the reference section we only list the works explicitly cited in the survey. Bibliographies in [ACGH], [FK] and [N] are extensive and should lead to further references.

References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Springer-Verlag, 1984.

[AD] E. Arbarello and C. DeConcini, *On a set of equations characterizing Riemann matrices*, Ann. of Math. **120** (1984), 119-140.

[FK] H. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, 1980.

[Gi] D. Gieseker, *Stable curves and special divisors*, Invent. Math. **66** (1982), 251-275.

[Gu] R. Gunning, *Lectures on Riemann Surfaces: Jacobi Varieties*, Princeton University Press, 1972.

[GH] P. Griffiths and J. Harris, *On the variety of special linear systems on a general curve*, Duke Math. J. **47** (1980), 233-272.

[H] J. Harris, *Curves and their moduli*, Proceedings of Symposia in Pure Math **46** (1987), 99-143

[K] C. Keem, *On the variety of special linear systems on an algebraic curve*, Math. Ann. **288** (1990), 309-322.

[N] M. Namba, *Families of Meromorphic Functions on Compact Riemann Surfaces*, Lecture Notes in Math. 767, Springer-Verlag, 1979

[S] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math. **83** (1986), 333-382.

[Y] K. Yang, *Meromorphic functions on a compact Riemann surface and associated complete minimal surfaces*, Proceedings of Amer. Math. Soc. **105** (1989), 706-711.

[Y2] K. Yang, *Complex Algebraic Geometry*, Marcel Dekker, New York, 1991.

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