J. Korean Math. Soc. **31** (1994), No. 4, pp. 559–568

# TRUNCATE PRODUCTS OF LATTICES

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A lattice is called *bounded* if it has both the least element and the largest element which are usually denoted by 0 (zero) and 1(unit), respectively. Recently, Bennett [2] defined the *rectangular product* of two bounded lattices $L_1$ and $L_2$ to be the set

$$\{(x,y) \in L_1 \times L_2 \mid x \neq 0, y \neq 0\} \cup \{(0,0)\}$$

with the order induced from the direct product $L_1 \times L_2$. With a minor change from this, the *truncate product* $L_1 \Diamond L_2$ of the two bounded lattices $L_1$ and $L_2$ is defined to be the set

$$\{(x,y) \in L_1 \times L_2 \mid x \neq 0 \neq y, x \neq 1 \neq y\} \cup \{(0,0),(1,1)\}$$

with the order also induced from the direct product $L_1 \times L_2$. In this paper, we investigate lattice properties preserved by taking truncate products. We assume throughout that every lattice is finite and bounded and we let $L^* = L - \{0,1\}$ for any bounded lattices $L$.

We first observe the following simple facts for any lattices $L_1$ and $L_2$ :

(i) $(x,y) \neq (0,0)$ in $L_1 \Diamond L_2$ if and only if $x \neq 0$ in $L_1$ and $y \neq 0$ in $L_2$.

(ii) $(x,y) \neq (1,1)$ in $L_1 \Diamond L_2$ if and only if $x \neq 1$ in $L_1$ and $y \neq 1$ in $L_2$.

(iii) $(x,y) \wedge (r,s) = (0,0)$ in $L_1 \Diamond L_2$ if and only if $x \wedge r = 0$ in $L_1$ or $y \wedge s = 0$ in $L_2$.

(iv) $(x,y) \vee (r,s) = (1,1)$ in $L_1 \Diamond L_2$ if and only if $x \vee r = 1$ in $L_1$ or $y \vee s = 1$ in $L_2$.

(v) Nonzero meets and nonunit joins agree on both $L_1 \Diamond L_2$ and $L_1 \times L_2$.

Received June 21, 1993.
This work was supported by KOSEF Grant 921-0100-010-2.

Let $L$ be a lattice. For elements $a > b$ in $L$, we write $a \succ b$ or $b \prec a$ (*a covers b* or *b is covered by a*) if $a \geq c > b$ implies $a = c$ for every element $c$ of $L$. An *atom* is any element which covers the least element and a *dual atom* is any element which is covered by the greatest element. Let us denote by $A(L)$ and $DA(L)$ the sets of all atoms and all dual atoms, respectively, of $L$. Then $A(L_1 \diamondsuit L_2) = A(L_1) \times A(L_2)$ and $DA(L_1 \diamondsuit L_2) = DA(L_1) \times DA(L_2)$, for any lattices $L_1$ and $L_2$. Furthermore, if $L_1$ or $L_2$ is isomorphic to $\mathbf{1}$ or $\mathbf{2}$, where $\mathbf{n}$ denotes an n-elements chain, then $L_1 \diamondsuit L_2$ is isomorphic to $\mathbf{2}$ and so, in this paper, we may assume that all lattices are not isomorphic to $\mathbf{1}$ and $\mathbf{2}$.

The truncate product of $\mathbf{2}^3$ and $\mathbf{2}^2$ is given in Figure 1. The resulting lattice is neither distributive nor modular. Furthermore, it is not uniquely complemented.

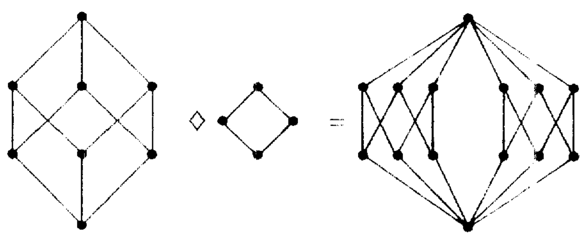

Figure 1

LEMMA 1. *Let $L_1$ and $L_2$ be lattices. Then the following statements hold:*

(i) *If $a$ is a complement of $c$ in $L_1^*$, then $(a, y)$ is a complement of $(c, z)$ in $L_1 \diamondsuit L_2$ for any $y, z \in L_2^*$.*
(ii) *If $b$ is a complement of $d$ in $L_2^*$, then $(x, b)$ is a complement of $(w, d)$ in $L_1 \diamondsuit L_2$ for any $x, w \in L_1^*$.*
(iii) *If $a \vee c = 1$ in $L_1$ and $b \wedge d = 0$ in $L_2$ or $a \wedge c = 0$ in $L_1$ and $b \vee d = 1$ in $L_2$ for $a, c \in L_1^*$ and $b, d \in L_2^*$, then $(a, b)$ is a complement of $(c, d)$ in $L_1 \diamondsuit L_2$.*

COROLLARY 1. *Let $L_1$ and $L_2$ be lattices and let $a \in L_1^*$ and $b \in L_2^*$. If $a$ is a complement of $c$ in $L_1$ and $b$ is a complement of $d$ in $L_2$, then $(c, y)$ and $(x, d)$ are complements of $(a, b)$ in $L_1 \diamondsuit L_2$ for any $x \in L_1^*$ and $y \in L_2^*$.*

LEMMA 2. *For any lattice $L$, $L \diamondsuit \mathbf{3} \cong L$.*

THEOREM 1. *The truncate product of two lattices is uniquely complemented if and only if one of its factors is uniquely complemented and the other is isomorphic to $\mathbf{3}$.*

*Proof.* The sufficiency of the condition follows easily from Lemma 2. To prove the necessity, let the truncate product of two lattices $L_1$ and $L_2$ be uniquely complemented. Let $a \in L_1^*$, $b \in L_2^*$ and let $(c, d)$ be a unique complement of $(a, b)$ in $L_1 \diamondsuit L_2$. Then $c \in L_1^*$, $d \in L_2^*$ and $(a, b) \vee (c, d) = (1, 1)$, $(a, b) \wedge (c, d) = (0, 0)$ in $L_1 \diamondsuit L_2$. Then we have four cases to consider :

(i) $a \vee c = 1$ and $a \wedge c = 0$ in $L_1$.
(ii) $b \vee d = 1$ and $b \wedge d = 0$ in $L_2$.
(iii) $a \vee c = 1$, $a \wedge c \neq 0$ in $L_1$ and $b \vee d \neq 1, b \wedge d = 0$ in $L_2$.
(iv) $a \vee c \neq 1$, $a \wedge c = 0$ in $L_1$ and $b \vee d = 1, b \wedge d \neq 0$ in $L_2$.

By symmetry we only treat the cases (i) and (iii).

CASE (I). $a \vee c = 1$ and $a \wedge c = 0$ in $L_1$.

Since $a$ is a complement of $c$ in $L_1$, $(c, y)$ is a complement of $(a, b)$ in $L_1 \diamondsuit L_2$ for any $y \in L_2^*$ by Lemma 1(i). Since $(c, d)$ is a unique complement of $(a, b)$ in $L_1 \diamondsuit L_2$, we have $y = d$ in $L_2$, and hence $| L_2^* | = 1$. Thus $L_2 \cong \mathbf{3}$ and so $L_1 \diamondsuit L_2 \cong L_1$ by Lemma 2. Hence $L_1$ is uniquely complemented, as required.

CASE (III). $a \vee c = 1$, $a \wedge c \neq 0$ in $L_1$ and $b \vee d \neq 1, b \wedge d = 0$ in $L_2$.

If $a \leq x < 1$ in $L_1$, then $x \vee c = a \vee c = 1$ in $L_1$. Since $b \wedge d = 0$ in $L_2$, it follows from Lemma 1(iii) that $(a, d)$ and $(x, d)$ are complements of $(c, b)$ in $L_1 \diamondsuit L_2$. As $L_1 \diamondsuit L_2$ is uniquely complemented, we have $a = x$ in $L_1$. Thus $a$ is a dual atom in $L_1$. Similarily, $c$ is also a dual atom in $L_1$ and so $|DA(L_1)| \geq 2$. Thus, choose a dual atom $z$ in $L_1$ with $z \neq a$, then $a \vee z = 1$ in $L_1$. Hence $(c, d)$ and $(z, d)$ are complements of $(a, b)$ in $L_1 \Diamond L_2$ by Lemma 1(iii). Since $L_1 \Diamond L_2$ is uniquely complemented, we have $z = c$ in $L_1$. Therefore, $L_1$ has exactly two dual atoms, that is, $DA(L_1) = \{a, c\}$.

Consider the element $(a \wedge c, b)$ in $L_1 \Diamond L_2$. Since $L_1 \Diamond L_2$ is uniquely complemented, there exists a unique element $(p, q)$ in $(L_1 \Diamond L_2)^*$ such that $(p, q) \vee (a \wedge c, b) = (1, 1)$ and $(p, q) \wedge (a \wedge c, b) = (0, 0)$. Since $P \neq 1$ in $L_1$ and $DA(L_1) = \{a, c\}$, we have $p \leq a$ or $p \leq c$ in $L_1$.

Now we claim that $p \wedge a \wedge c \neq 0$ in $L_1$. If $p \leq a$ and $p \leq c$, then clearly $p \wedge a \wedge c = p \neq 0$ in $L$. Next we consider the case when $p \leq a$ and $p \not\leq c$. Suppose that $p \wedge a \wedge c = 0$ in $L_1$. Since $p \leq a$, $p \not\leq c$ in $L_1$ and $c \in DA(L_1)$, $p \wedge c = p \wedge a \wedge c = 0$ and $p \vee c = 1$ in $L_1$, and hence $p$ is a complement of $c$ in $L_1$. Thus $(p, b)$ is a complement of $(c, d)$ in $L_1 \Diamond L_2$ by Lemma 1(i). Since $(a, b)$ is a unique complement of $(c, d)$ in $L_1 \Diamond L_2$, we have $p = a$ in $L_1$. Hence $p \wedge a \wedge c = a \wedge c \neq 0$ in $L_1$, which is a contradiction. Hence $p \wedge a \wedge c \neq 0$ in $L_1$. The case when $p \not\leq a$ and $p \leq c$ is similarly treated.

By the preceding claim, $(p, q) \wedge (a \wedge c, b) = (0, 0)$ implies that $q \wedge b = 0$ in $L_2$. Since $(p, q) \vee (a \wedge c, b) = (1, 1)$ and $DA(L_1) = \{a, c\}$, we have $p \vee (a \wedge c) \leq a \prec 1$ or $p \vee (a \wedge c) \leq c \prec 1$, and so $q \vee b = 1$ in $L_2$. Hence $b$ has a complement $q$ in $L_2^*$. By duality, we obtain $A(L_2) = \{b, d\}$ and so $a$ has a complement $r$ in $L_1^*$. By Corollary 1, $(a, q)$ $(r, q)$ and $(r, b)$ are distinct complements of $(a, b)$ in $L_1 \Diamond L_2$, which is a contradiction.

COROLLARY 2. *The truncate product of two lattices is isomorphic to* $\mathbf{2}^n$ *if and only if one of its factors is isomorphic to* $\mathbf{2}^n$ *and the other is isomorphic to* $\mathbf{3}$.

A lattice $L$ is said to be *(upper) semimodular* (or satisfy the *upper covering property*) if for any elements $a, b$ and $c$, of $L$ $a \prec b$ implies that $a \vee c = b \vee c$ or $a \vee c \prec b \vee c$. The *lower semimodularity* (or *lower covering property*) is defined dually. We observe that a lattice is semimodular if and only if $x \succ x \wedge y$ implies $x \vee y \succ y$. In [1], it is shown that the rectangular product of lattices $L_1$ and $L_2$ is semimodular if and only if $L_1$ and $L_2$ are semimodular with $|A(L_1)| = 1$ or $|A(L_2)| = 1$. But this condition does not apply for the truncate product ( Figure 2 ).

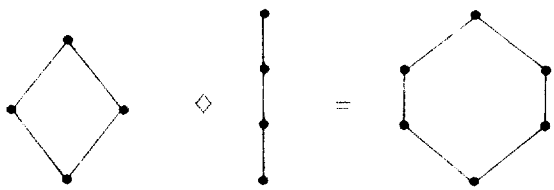

Figure 2

The *length* of an $n$-element chain $\mathbf{n}$ is defined to be $n-1$. More generally, the *length* $l(P)$ of an ordered set $P$ is defined as the supremum of the lengths of chains in $P$. In an ordered set $P$ of finite length with the least element 0, the *height* $h(x)$ of an element $x \in P$ is $l([0,x])$, where $[0,x] = \{y \in P | y \leq x\}$. If $P$ has the greatest element 1, then clearly $h(1) = l(P)$.

THEOREM 2. *The truncate product of lattices $L_1$ and $L_2$ is semimodular if and only if $L_1$ and $L_2$ are semimodular and one of the following conditions holds:*

(i) *$L_1$ and $L_2$ are isomorphic to $M_r$ and $M_s$, respectively, for some positive integers $r$ and $s$, where $M_n$ is the lattice of length 2 with $n$ atoms.*

(ii) *$L_1$ or $L_2$ is isomorphic to $\mathbf{3}$.*

(iii) *$|DA(L_i)| = 1$ for $i = 1,2$ and $|A(L_1)| = 1$ or $|A(L_2)| = 1$.*

*Proof.* ($\Longrightarrow$) Suppose that $L_1 \diamondsuit L_2$ is semimodular such that $L_1 \diamondsuit L_2 \not\cong M_n$ for any positive integer $n$ and neither $L_1$ nor $L_2$ is isomrphic to $\mathbf{3}$. Then we shall show that both $L_1$ and $L_2$ are semimodular, $|DA(L_i)| = 1 (i = 1,2)$ and $|A(L_1)| = 1$ or $|A(L_2)| = 1$.

CLAIM 1. Both $L_1$ and $L_2$ are semimodular.

Suppose that $L_1$ or $L_2$ is not semimodular. We may assume without loss of generality that $L_2$ is not semimodular. Then, there are elements $b, d \in L_2^*$ such that $b$ covers $b \wedge d$ in $L_2$ and $b \vee d$ does not cover $d$ in $L_2$. Pick an element $x$ in $L_2$ such that $d < x < b \vee d$ in $L_2$. Consider the elements $(p,b) \wedge (p,d)$, $(p,b)$, $(p,d)$ and $(p,b) \vee (p,d)$ in $L_1 \diamondsuit L_2$ for some atom $p$ in $L_1$. Since $b \succ b \wedge d$ in $L_2$, we have $(p,b) \succ$

$(p,b) \wedge (p,d)$ in $L_1 \Diamond L_2$. Furthermore, we know that if $b \vee d = 1$ in $L_2$ then $(p,b) \vee (p,d) = (1,1)$ in $L_1 \Diamond L_2$ and that if $b \vee d \neq 1$ in $L_2$ then $(p,b) \vee (p,d) = (p, b \vee d)$ in $L_1 \Diamond L_2$. In either case, we have $(p,d) < (p,x) < (p,b) \vee (p,d)$ in $L_1 \Diamond L_2$, and so $(p,b) \vee (p,d)$ does not cover $(p,d)$ in $L_1 \Diamond L_2$, which is a contradiction.

CLAIM 2. $|DA(L_1)| = 1$ and $|DA(L_2)| = 1$.

Suppose that $|DA(L_1)| \geq 2$ or $|DA(L_2)| \geq 2$. We may assume without loss of generality that $|DA(L_2)| \geq 2$. Note that $L_i (i = 1,2)$ does not isomorphic to **1**, **2** and $M_n$ for any positive integer $n$. Thus $l(L_1) \geq 3$. Since $|DA(L_2)| \geq 2$ and so there are distinct two dual atoms $b, d$ in $L_2$. Pick $y$ such that $1 \succ b \geq y \succ b \wedge d$. Since $l(L_1) \geq 3$, there are elements $a$ and $c$ in $L_1$ such that $0 \prec c < a \prec 1$. Since $b \geq y \succ y \wedge d = b \wedge d$ in $L_2$, $y$ and $d$ are incomparable elements in $L_2$. Furthermore, since $d$ is a dual atom in $L_2$, we have $y \vee d = 1$ in $L_2$. If $y \wedge d = 0$ in $L_2$, then $y \succ y \wedge d = 0$ in $L_2$ and so $y \in A(L_2)$. Hence $(c,y) \succ (c,y) \wedge (c,d) = (0,0)$ in $L_1 \Diamond L_2$. But $(c,y) \vee (c,d) = (1,1) \succ (a,d) > (c,d)$ in $L_1 \Diamond L_2$, which is a contradiction to the semimodularity of $L_1 \Diamond L_2$. If $y \wedge d \neq 0$ in $L_2$, then $(c,y) \succ (c, y \wedge d) = (c,y) \wedge (c,d)$ in $L_1 \Diamond L_2$. But $(c,y) \vee (c,d) = (1,1) \succ (a,d) > (c,d)$ in $L_1 \Diamond L_2$, which is also a contradiction to the semimodularity of $L_1 \Diamond L_2$

CLAIM 3. $|A(L_1)| = 1$ or $|A(L_2)| = 1$.

Suppose that $|A(L_1)| \geq 2$ and $|A(L_2)| \geq 2$. Then, there are two distinct atoms $p_i, q_i$ in each $L_i$ for $i = 1,2$. Then $(p_1, p_2) \succ (0,0) = (p_1,p_2) \wedge (q_1,q_2)$ in $L_1 \Diamond L_2$. By Claim 2, $(p_1,p_2) \vee (q_1,q_2) \neq (1,1)$ in $L_1 \Diamond L_2$ and so $(p_1,p_2) \vee (q_1,q_2) = (p_1 \vee q_1, p_2 \vee q_2) \succ (q_1, p_2 \vee q_2) \succ (q_1,q_2)$ in $L_1 \Diamond L_2$. Thus, $(p_1,p_2) \vee (q_1,q_2)$ does not cover $(q_1,q_2)$ in $L_1 \Diamond L_2$, which is a contradiction.

($\Longleftarrow$) Suppose that $L_1$ and $L_2$ are semimodular. Since other cases are trivial, we only assume that $|DA(L_i)| = 1$ for $i = 1, 2$ and $|A(L_1)| = 1$. Let $(a,b) \succ (a,b) \wedge (c,d)$ in $L_1 \Diamond L_2$. We shall show that $(a,b) \vee (c,d) \succ (c,d)$ in $L_1 \Diamond L_2$. We have three cases to consider.

CASE (1). If $a \wedge c = 0$ in $L_1$, then $(a,b) \succ (a,b) \wedge (c,d) = (0,0)$ in $L_1 \Diamond L_2$ and so $a \in A(L_1)$ and $b \in A(L_2)$. Since $A(L_1) = \{a\}$ and $a \wedge c = 0$ in $L_1$, we have $c = 0$ in $L_1$. Hence, $(c,d) = (0,0)$ in $L_1 \Diamond L_2$ and so $(a,b) \vee (c,d) = (a,b) \succ (0,0) = (c,d)$ in $L_1 \Diamond L_2$.

CASE (II). If $a \wedge c \neq 0$ in $L_1$ and $b \wedge d = 0$ in $L_2$, then $(a,b) \succ (a,b) \wedge (c,d) = (0,0)$ in $L_1 \Diamond L_2$ and again $a \in A(L_1)$ and $b \in A(L_2)$. Since $A(L_1) = \{a\}$ and $a \wedge c \neq 0$, we have $a \leq c$ in $L_1$. As $b \succ 0 = b \wedge d$ and $L_2$ is semimodular, $b \vee d \succ d$ in $L_2$. Now, since $|DA(L_2)| = 1$, we have $b \vee d \neq 1$ in $L_2$ and so $(a,b) \vee (c,d) = (c, b \vee d) \succ (c,d)$ in $L_1 \Diamond L_2$.

CASE (III). If $a \wedge c \neq 0$ in $L_1$ and $b \wedge d \neq 0$ in $L_2$, then $(a,b) \succ (a,b) \wedge (c,d) = (a \wedge c, b \wedge d)$ in $L_1 \Diamond L_2$. Hence, $a = a \wedge c$ in $L_1$ and $b \succ b \wedge d$ in $L_2$, or $a \succ a \wedge c$ in $L_1$ and $b = b \wedge d$ in $L_2$. Since $L_1$ and $L_2$ are semimodular, $a \leq c$ in $L_1$ and $b \vee d \succ d$ in $L_2$, or $a \vee c \succ c$ in $L_1$ and $b \leq d$ in $L_2$. Observe that $c \neq 1 \neq d$ in either case. If $a \leq c$ in $L_1$ and $b \vee d \succ d$ in $L_2$, then $b \vee d \neq 1$ in $L_2$. For, otherwise we have $b = 1$ and so $a = 1$, whence $c = 1$, contradicting to the preceding observation. Similary, if $b \leq d$ and $a \vee c \succ c$, then $a \vee c \neq 1$. Now, since $(a,b) \vee (c,d) = (c, b \vee d)$ in $L_1 \Diamond L_2$ or $(a,b) \vee (c,d) = (a \vee c, d)$ in $L_1 \Diamond L_2$, we have $(a,b) \vee (c,d) \succ (c,d)$ in $L_1 \Diamond L_2$.

Let $L$ be a lattice of finite length. It is well known that $L$ is modular if and only if $L$ is both upper and lower semimodular. By Theorem 2 and duality, the following corollaries are immediately obtained.

COROLLARY 3. *The truncate product of lattices $L_1$ and $L_2$ is modular if and only if $L_1$ and $L_2$ are modular and one of the following conditions holds:*

(i) *$L_1$ and $L_2$ is isomorphic to $M_r$ and $M_s$, respectively, for some positive integers $r$ and $s$.*
(ii) *$L_1$ or $L_2$ is isomorphic to $\mathbf{3}$.*
(iii) *$|DA(L_i)| = 1$ and $|A(L_i)| = 1$ for $i = 1, 2$.*

COROLLARY 4. *The truncate product of lattices $L_1$ and $L_2$ is distributive if and only if $L_1$ and $L_2$ are distributive and one of the following conditions holds :*

(i) *$L_1$ or $L_2$ is isomorphic to $\mathbf{3}$.*
(ii) *$|DA(L_i)| = 1$ and $|A(L_i)| = 1$ for $i = 1, 2$.*

Finally, we list a series of propositions whose proofs are quite straightforward.

An ordered set $P$ with 0 is called *graded* if for $x, y \in P$, $x \leq y$ and $h(x) + 1 = h(y)$ if and only if $x \prec y$, and is said to satisfy the

*Jordan-Dedekind Chain Condition* if all maximal chains between the same endpoints have the same finite length. Observe that an ordered set $P$ with 0 is graded if and only if every interval of $P$ is of finite length and satisfies the Jordan-Dedekind Chain Condition.

PROPOSITION 1. *The truncate product of lattices $L_1$ and $L_2$ is graded if and only if both $L_1$ and $L_2$ are.*

We call a lattice *atomic* when every nonzero element is the join of the atoms below it.

PROPOSITION 2. *The truncate product of lattices $L_1$ and $L_2$ is atomic if and only if both $L_1$ and $L_2$ are.*

We call a lattice $L$ *biatomic* when every nonzero element is the join of the atoms below it and whenever $p$ is an atom of $L$ with $p \leq a \vee b$ there are atoms $a_1 \leq a$ and $b_1 \leq b$ with $p \leq a_1 \vee b_1$. In [2], it is shown that a rectangular product of lattices is biatomic exactly when both of its factors are biatomic. But the biatomicity need not be preserved by taking truncate product, as Figure 1 shows. A pair of nonzero elements $a$ and $b$ of a lattice $L$ is called a *biatomic pair* if whenever $p$ is an atom of $L$ with $p \leq a \vee b$ there are atoms $a_1 \leq a$ and $b_1 \leq b$ with $p \leq a_1 \vee b_1$. We then define an atomic lattice to be *weakly biatomic* if for any nonzero elements $a$ and $b$ with $a \vee b \neq 1$ form a biatomic pair.

PROPOSITION 3. *The truncate product of lattices $L_1$ and $L_2$ is weakly biatomic if and only if both $L_1$ and $L_2$ are.*

PROPOSITION 4. *If the truncate product of lattices $L_1$ and $L_2$ is biatomic, then $L_1$ and $L_2$ are also biatomic.*

A lattice $L$ is said to be *join-semidistributive* if $a \vee b = a \vee c$ implies that $a \vee b = a \vee (b \wedge c)$ for all $a, b, c \in L$. As we see in Figure 1, even if $L_1$ and $L_2$ are join-semidistributive, $L_1 \Diamond L_2$ need not be join-semidistributive. Furthermore, in [1], if $L_1$ and $L_2$ are biatomic and join-semidistributive and if they satisfy the ascending chain condition, then the rectangular product of $L_1$ and $L_2$ is join-semidistributive. Although $L_1 = L_2 = \mathbf{2}^2$ satisfies the hypotheses, $L_1 \Diamond L_2 = M_4$ is not join-semidistributive. But the following proposition shows that the converse is true for truncate products.

PROPOSITION 5. *If the truncate product of lattices $L_1$ and $L_2$ is join-semidistributive, then $L_1$ and $L_2$ are also join-semidistributive.*

We know that, in any lattice $L$, the following conditions are equivalent:

(i) For any elements $x_1, x_2, \cdots, x_n \in L$, if $x_i \wedge x_j \neq 0$ for all $i \neq j$, then $x_1 \wedge x_2 \wedge \cdots \wedge x_n \neq 0$ in $L$.

(ii) For any elements $x, y, z$ in $L$, if each of $x \wedge y$, $y \wedge z$ and $z \wedge x$ is nonzero elements, then $x \wedge y \wedge z$ is nonzero.

A lattice $L$ is said to satisfy the *Chinese Remainder Theorem*(C R T) if and only if the condition (i) (and hence (ii)) holds in it. In [1], we have shown that the rectangular product of $L_1$ and $L_2$ satisfies the C R T if and only if both $L_1$ and $L_2$ do. Since nonzero meets and nonunit joins agree on both the truncate product and the rectangular product, we obtain the following proposition.

PROPOSITION 6. *The truncate product of lattices $L_1$ and $L_2$ satisfies the C R T if and only if both $L_1$ and $L_2$ do.*

A lattice $L$ satisfies the ***strict Chinese Remainder Theorem*** (strict C R T) if $L$ is atomic and, for any atoms $p, q, r$ in $L$, $M(p,q,r) = (p \vee q) \wedge (q \vee r) \wedge (r \vee p)$ is an atom in $L$. Note that the strict C R T implies the C R T, but the converse may not hold (see $M_n$). In [1], it is shown that the rectangular product of $L_1$ and $L_2$ satisfies the strict C R T if and only if both $L_1$ and $L_2$ do. But the truncate product of $\mathbf{2}^2$ and $M_n$ is $M_{2n}$ which does not satisfy the strict C R T. However the converse is true.

PROPOSITION 7. *If the truncate product of lattices $L_1$ and $L_2$ satisfies the strict C R T, then both $L_1$ and $L_2$ do.*

An atomic lattice $L$ satisfies the ***switching condition*** if $p, q \leq r \vee s$ and $p \leq r \vee q$ imply that $q \leq s \vee p$ for any atoms $p, q, r, s$ of $L$. In [1], it is shown that the rectangular product of $L_1$ and $L_2$ satisfies the switching condition if and only if both $L_1$ and $L_2$ do. We also have the following proposition.

PROPOSITION 8. *The truncate product of lattices $L_1$ and $L_2$ satisfies the switching condition if and only if both $L_1$ and $L_2$ do.*

## References

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