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POSITIVE SOLUTIONS FOR PREDATOR–PREY EQUATIONS WITH NONLINEAR DIFFUSION RATES

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1. Introduction

In this paper, we will investigate the existence of positive solutions to the predator-prey interacting system

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u,v) & \text{in } \Omega \\ -\psi(x,v)\Delta v = vg(x,u,v) & \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0. & \end{cases} \tag{1}$$

in a bounded region Ω in $\mathbf{R}^n$ with smooth boundary, where φ and ψ are strictly positive functions, serving as nonlinear diffusion rates, and κ, $\sigma > 0$ are constants. Assume that the growth rates f, g are C^1 monotone functions. The variables u, v may represent the population densities of the interacting species in problems from ecology, microbiology, etc.

In [12], the solutions of the equations

$$\begin{cases} -\Delta u = uM(x,u,v) & \\ -\Delta v = vN(x,u,v) & \text{in } \Omega \end{cases}$$

under the boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ are investigated in the competition and symbiosis cases, using the monotone-iteration scheme. For the predator-prey case, the variational approach was used in [4] for the first time.

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In [5], [6] and [7], necessary and sufficient conditions for the existence of positive solutions of the elliptic system

$$\begin{cases} -d_1\Delta u = uM(u,v) \\ -d_2\Delta v = vN(u,v) \\ u = v = 0 \qquad \text{on } \partial\Omega \end{cases}$$

with constant diffusion rates have been established for all the possible cases of monotonicities associated with the functions M and N. For those works, index theory ([5], [7]) and decomposed operators ([6]) have been employed to prove the existence of positive solutions.

Therefore, the problems of the existence of positive solutions for the nonlinear interacting systems with constant diffusion rates are to a large degree solved. However, in many chemical and biological systems, the diffusion rates indeed depend on the densities u and v. Thus we need to study the reaction-diffusion systems with nonlinear diffusion rate from this point of view.

To attack our problem, we first give some *a priori* estimates on solutions to the system (1) to have the compactness of the corresponding operators for the nonlinear elliptic equations. We then handle the linearization to equations.

In section 2, we provide a sequence of lemmas which will be used to prove the results in section 3. In section 3, we will give sufficient and necessary conditions for existence of positive solutions of (1) for the predator-prey interaction. The existence of positive solutions can be characterized by the spectral property of a certain operator of Schrödinger type.

2. Preliminaries

In this paper we will consider problems in the space $\mathbf{X} = C(\bar{\Omega})$, where Ω is a bounded region in $\mathbf{R}^n$ and let $r(T)$ denote the spectral radius of a linear operator T.

Let $f = f(x,\xi)$. Then $f \in F$ if and only if $f \in C(\bar{\Omega} \times \mathbf{R}^+)$ and

(F1) $f \in C^1$ in ξ, $f_\xi(x,\xi) < 0$ in $\Omega \times \mathbf{R}^+$, and for some $N \in \mathbf{R}^+$, $|f_\xi(x,\xi)| \le N$ where $(x,\xi) \in \Omega \times [0,c_0]$.

(F2) $f(x,0) > 0$ and $f(x,\xi) < 0$, where $(x,\xi) \in \Omega \times (c_0,\infty)$ for some constant $c_0 > 0$.

(F3) $f(x,\xi)$ is concave down on the set of (x,ξ) where $f(x,\xi) < 0$.

Let $\varphi = \varphi(x,\xi)$. Then $\varphi \in G$ if and only if $\varphi \in C(\bar{\Omega} \times \mathbf{R}^+)$ and φ is C^1-function in ξ which in addition satisfies

(G1) $\varphi(x,\xi) \geq \delta > 0$ for some constant δ and $\xi \in \mathbf{R}^+$, $x \in \Omega$.

(G2) φ is nondecreasing and concave down in $\xi \in \mathbf{R}^+$.

LEMMA 1. *Let $f \in F$ and $\varphi \in G$. Then $\frac{f(x,u)}{\varphi(x,u)}$ is decreasing in u for $x \in \Omega$.*

Proof. We consider bounded regions Ω_j, $j = 1,2,3$, $\Omega_1 = \{x \in \Omega : f(x,u_1) \geq 0 \text{ and } f(x,u_2) > 0\}$, $\Omega_2 = \{x \in \Omega : f(x,u_1) < 0 \text{ and } f(x,u_2) \geq 0\}$, $\Omega_3 = \{x \in \Omega : f(x,u_1) < 0 \text{ and } f(x,u_2) \leq 0\}$. Then $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \Omega_3$.

For $x \in \Omega_1$ or $x \in \Omega_2$, it is easy to see the monotonicity of $\frac{f}{\varphi}$. Suppose $x \in \Omega_3$. Note that $\frac{\varphi(x,u_1)}{u_1} \leq \frac{\varphi(x,u_2)}{u_2}$ and $\frac{f(x,u_1)}{u_1} \leq \frac{f(x,u_2)}{u_2}$ by the assumptions (F3) and (G2). Since $f(x,u_1) \leq f(x,u_2) \leq 0$, one must have $\frac{f(x,u_1)}{u_1} \cdot \frac{\varphi(x,u_2)}{u_2} \leq \frac{f(x,u_2)}{u_2} \cdot \frac{\varphi(x,u_1)}{u_1}$. Therefore we have that for $x \in \Omega_3$, $\frac{f(x,u_1)}{\varphi(x,u_1)} - \frac{f(x,u_2)}{\varphi(x,u_2)} = \frac{f(x,u_1)/u_1}{\varphi(x,u_1)/u_1} - \frac{f(x,u_2)/u_2}{\varphi(x,u_2)/u_2} \leq 0$. Thus for $x \in \Omega$, $\frac{f(x,u)}{\varphi(x,u)}$ is decreasing in u.

LEMMA 2. *Let $P > 0$ be a constant. Assume $\varphi \in G$. Also let $0 \not\equiv h \geq 0$, $h \in C^\alpha(\bar{\Omega})$ where $\alpha \in (0,1)$. Consider*

$$\begin{cases} -\varphi(x,u)\Delta u + Pu = h \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Then (2) has a unique positive solution $u \in C^{2,\alpha}(\Omega)$. Moreover, the solution operator S such that $u = Sh$ is compact in $\mathbf{K}$ or $C(\bar{\Omega})$, where $\mathbf{K}$ is the positive cone of $C(\bar{\Omega})$.

Proof. The positiveness of solutions to (2) for $h \geq 0$ follows from the strong maximum principle. First we show that the nonnegative solution to (2) is unique when $0 \leq h \in C^\alpha(\bar{\Omega})$. Let u and v be two distinct nonnegative solutions of (2). Without loss of generality, let $\min_{x\in\bar{\Omega}}(u(x)-v(x)) < 0$. Let $u(x_0)-v(x_0) = \min_{x\in\bar{\Omega}}(u(x)-v(x)) < 0$. Assume $x_0 \in \partial\Omega$. Then by the minimality of $u - v$ at x_0, we have $\frac{\partial(u-v)(x_0)}{\partial n} \leq 0$ and $\kappa[u(x_0) - v(x_0)] < 0$. Thus the boundary condition becomes $[\frac{\partial u(x_0)}{\partial n} - \frac{\partial v(x_0)}{\partial n}] + \kappa[u(x_0) - v(x_0)] < 0$, which is a contradiction. Thus $x_0 \notin \partial\Omega$. Since u and v are solutions of (2), we have

$$\begin{aligned} &-\varphi(x,u)\varphi(x,v)\Delta(u-v) + P(u\varphi(x,v) - v\varphi(x,u)) \\ =&h(x)(\varphi(x,v) - \varphi(x,u)) \end{aligned} \tag{3}$$

Observe that at $x_0 \in \Omega$, $-\varphi(x_0, u(x_0))\varphi(x_0, v(x_0))\Delta(u(x_0) - v(x_0)) \leq 0$ by the minimizing property of x_0. Also we have $P \cdot [u(x_0)\varphi(x_0, v(x_0)) - v(x_0)\varphi(x_0, u(x_0))] < 0$ since the nondecreasing function $\varphi > 0$ is concave down. Hence the left side of (3) is negative. Since $v(x_0) > u(x_0)$ and $h(x_0) \geq 0$, the right side of (3) is nonnegative, which is a contradiction. Therefore we must have $u = v$.

Next we shall prove the existence of a solution. First we can show the *a priori bound* in a space $C^{2,\alpha}(\bar{\Omega})$ of every solution to (2), where $\alpha \in (0,1)$ and then find a fixed point of the following equation

$$\begin{cases} -\Delta u = \frac{h - Pv}{\varphi(x,v)} \\ \frac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{4}$$

Let u be a solution of equation (2) for $x \in \Omega$, i.e., u is a fixed point of equation (4). Denote Green's function under the Robin boundary condition $\frac{\partial \cdot}{\partial n} + \kappa \cdot = 0$ by G_R. Then

$$u(x) = \int_\Omega G_R(x,y)\left(\frac{h - Pu(y)}{\varphi(y, u(y))}\right)dy. \tag{5}$$

Since $\|Pu\|_\infty \leq \|h\|_\infty$ by a general maximum principle, we can estimate:

$$\begin{aligned} &\int_\Omega \left| G_R \frac{h - Pu}{\varphi} \right| \\ \leq &\|G_R\|_{L^{n/n-1}} \|h - Pu\|_{L^n} \cdot \delta^{-1} < K_1 2\|h\|_\infty \delta^{-1} = K_2 \|h\|_\infty. \end{aligned} \tag{6}$$

$$\begin{aligned} &\int_\Omega \left| \frac{\partial G_R}{\partial x_i} \frac{h - Pu}{\varphi} \right| \\ \leq &\frac{N}{\delta^{-1}} \|h - Pu\|_{L^m} \int_\Omega \frac{1}{r^{\frac{(n-1)m}{m-1}}} < 2\|h\|_\infty \delta^{-1} \tilde{N} = K\|h\|_\infty \end{aligned} \tag{7}$$

where $\tilde{N} = \int \frac{1}{r^{\frac{(n-1)m}{m-1}}}$ and $r = \|x - y\|$. Thus (6), (7) imply

$$\|D_{x_i} u\|_\infty \leq (K\|h\|_\infty). \tag{8}$$

So applying the elliptic regularity to (4) and the fact $\|\nabla u\|_\infty \leq M\|h\|_\infty$, we have $u \in W^{2,m}(\Omega)$ where $m > n$ and by (8)

$$\|u\|_{W^{2,m}} \leq C\|h\|_\infty \tag{9}$$

for some constant C. Since $m > n$, the Sobolev imbedding theorem implies

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq M_1\|h\|_\infty \tag{10}$$

where some $\alpha \in (0,1)$. On the other hand, the Schauder estimate ([2]) gives

$$\|u\|_{C^{2,\alpha}} \leq C_1(n,\alpha)(\|u\|_\infty + \delta^{-1} \cdot \|h - Pu\|_{C^\alpha}). \tag{11}$$

Therefore $\|u\|_{C^{2,\alpha}}$ is also bounded.

Let $v \in C^{2,\alpha}(\bar{\Omega})$ and define $T : C^{2,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ such that $u = Tv$ is a solution of (4). Then T is continuous, compact and bounded. Therefore the Schauder fixed point theorem provides a fixed point u such that $Tu = u$.

The compactness of the solution operator S in $C(\bar{\Omega})$ such that $u = Sh$ follows from (9), (10) and (11) by the Ascoli-Arzéla theorem.

COROLLARY. *Let $P > 0$ and $\tau > 0$ be constants. Suppose $a(x) \in C^1(\bar{\Omega})$ and $a(x) \geq \delta_0 > 0$, Consider for $\alpha \in (0, 1)$,*

$$\begin{cases} -a(x)\Delta u + Pu = h(x), & h \in C^\alpha(\bar{\Omega}) \\ \dfrac{\partial u}{\partial n} + \tau u = 0, & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Then (12) has a unique solution u in $C^2(\bar{\Omega})$ and the solution operator T such that $u = Th$ is a compact operator in $\mathbf{X} = C(\bar{\Omega})$.

The following is well-known. We record this to use in the proof of other results.

LEMMA 3. *Suppose $a \in C^1(\bar{\Omega})$ and $a(x) \geq \delta_0 > 0$ and $b(x) \in L^\infty(\Omega)$. Let $\tau > 0$ be a constant. Then there exists $u > 0$, $u \in C^2(\bar{\Omega})$, such that for a unique $\lambda_1 > 0$*

$$\begin{cases} -a(x)\Delta u + b(x)u = \lambda_1 u \\ \dfrac{\partial u}{\partial n} + \tau u = 0 \qquad on\ \partial\Omega. \end{cases} \tag{13}$$

Moreover, λ_1 is increasing in $a(x)$ and a ratio $\frac{b(x)}{a(x)}$.

Let $\mathbf{X}$ be a Banach space and let F be a strongly positive nonlinear compact operator on $\mathbf{X}$ such that $F(0) = 0$.

LEMMA 4. *Assume $F'(0)$ exists with $r(F'(0)) > 1$. If there is no $\mu \in (0,1]$ in any neighborhood of which the equation $u = \mu F u$ has a solution u as $\|u\| \to \infty$, then F has a positive fixed point u such that $Fu = u$ in the positive cone $\mathbf{K}$ of $\mathbf{X}$.*

Proof. See Theorem 13.2 in [1].

LEMMA 5. *Let T be a compact positive linear operator on an ordered Banach space. Let $u > 0$ be a positive element.*
(i) If $Tu > u$, then $r(T) > 1$
(ii) If $Tu < u$, then $r(T) < 1$
(iii) If $Tu = u$, then $r(T) = 1$.

Proof. See Lemma 2.3 in [5].

LEMMA 6. *Let $a(x) \geq \delta_0 > 0$ and $b(x) \in L^\infty(\Omega)$. Also let P be positive constant such that $P + b(x) > 0$ for $x \in \Omega$. Then*
(i) $\lambda_1(a(x)\Delta + b(x)) > 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] > 1$
(ii) $\lambda_1(a(x)\Delta + b(x)) < 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] < 1$
(iii) $\lambda_1(a(x)\Delta + b(x)) = 0 \Leftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] = 1$
where λ_1 is the first eigenvalue under Robin boundary condition.

Proof. Let $\phi > 0$ be the eigenfunction corresponding to the eigenvalue $\lambda_1(a(x)\Delta + b(x))$. Then $(-a(x)\Delta + P)\phi = (P + b(x))\phi - \lambda_1(a(x)\Delta + b(x))\phi$. So one can see that $(-a(x)\Delta + P)\phi >, =, < (P + b(x))\phi$ depending on the sign of $\lambda_1(a(x)\Delta + b(x))$. Let $T := (-a(x)\Delta + P)^{-1}(P + b(x))$. Then T is a positive compact operator under Robin boundary condition in $C(\bar{\Omega})$ by Corollary. Therefore one may apply Lemma 5 to get the results.

Consider the following equation,

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Next we shall linearize the equation (14) at $u = 0$. Use Lemma 2 to define the solution operator S in $C(\bar{\Omega})$ by $Su = \tilde{u}$, where $\tilde{u}$ is the unique solution of

$$\begin{cases} -\varphi(x,\tilde{u})\Delta\tilde{u} + P\tilde{u} = uf(x,u) + Pu \\ \dfrac{\partial \tilde{u}}{\partial n} + \kappa\tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Also we define the solution operator S_L of linearization by $S_L w = v$, where v is the unique solution of

$$\begin{cases} -\varphi(x,0)\Delta v + Pv = wf(x,0) + Pw \\ \dfrac{\partial v}{\partial n} + \kappa v = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

LEMMA 7. *The operator S is Frechét differentiable at $u = 0$, and $S'(0) = S_L$.*

Proof. Let $\tilde{u} = Su$ be the unique solution of (15) and v be the solution to (16). Assume that $\|u\|_\infty$ is small. Then from (15) and (16), we have

$$\begin{aligned} &-\varphi(x,0)\Delta(\tilde{u}-v) + P(\tilde{u}-v) = \varphi(x,0)u\left[\frac{f(x,u)}{\varphi(x,\tilde{u})} - \frac{f(x,0)}{\varphi(x,0)}\right] \\ &+\varphi(x,0)Pu\left[\frac{1}{\varphi(x,\tilde{u})} - \frac{1}{\varphi(x,0)}\right] - \varphi(x,0)P\tilde{u}\left[\frac{1}{\varphi(x,\tilde{u})} + \frac{1}{\varphi(x,0)}\right] :\equiv h \end{aligned} \tag{17}$$

$$\frac{\partial(\tilde{u}-v)}{\partial n} + \kappa(\tilde{u}(x) - v(x)) = 0 \qquad \text{on } \partial\Omega.$$

It is easy to see that $\|\tilde{u}\|_\infty = O(\|u\|_\infty)$, and $\|h\|_\infty = o(\|u\|_\infty)$ when u is small. Thus $\|\tilde{u} - v\|_\infty = O(\|h\|_\infty) = o(\|u\|_\infty)$.

Let $\mathbf{K} = C(\bar{\Omega})^+$ be the positive cone of the ordered Banach space $C(\bar{\Omega})$. We define the ordered interval $[[u_1, u_2]] := \{u \in C(\bar{\Omega}) : u_1 \leq u \leq u_2 \text{ for } u_1, u_2 \in C(\bar{\Omega})\}$. In the next two lemmas, $\lambda_1(A)$ denotes the first eigenvalue of an operator A under the boundary condition $\frac{\partial u}{\partial n} + \kappa u = 0$.

LEMMA 8. *Let $f \in F$ and $\varphi \in G$.*

(i) If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) > 0$, then the equations (14) have a unique positive solution in $C^2(\bar{\Omega})$.

(ii) If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) \leq 0$, then $u \equiv 0$ is the only nonnegative solution of (14).

Proof. (i) Suppose $\lambda_1(\varphi(x,0)\Delta + f(x,0)) > 0$. If u is a positive solution of (14), then $0 \leq u < c_0$ by the general maximum principle.

Choose $P > 2\max\{\sup_{\bar{\Omega}\times[0,c_0]} |f(x,\xi)|, c_0 L\}$. Let $[[0, c_0]]$ denote the order interval in $C(\bar{\Omega})$. We define an operator $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ by $Au = S(uf(x,u) + Pu)$ where S is again defined as the solution operator of the equation (2). Then A is an increasing strongly positive compact operator from $[[0, c_0]]$ to $C(\bar{\Omega})$ by Lemma 2. Note that u is a positive solution of (14) if and only if u is a fixed point of the operator A. It is easy to see that $\hat{u} \equiv c_0$ is an upper solution of the equation of (14). So we have $A'(\hat{u}) \leq \hat{u}$. Also note that $\bar{u} \equiv 0$ is a solution of (14) and Lemma 7 implies that $A'(\bar{u}) = A'(0) = (-\varphi(x,0)\Delta + P)^{-1}[f(x,0) + P]$. Thus we have $r(A'(0)) > 1$ by Lemma 6. Now applying Theorem 7.6 in [1], we have a maximal solution u in $[[0, c_0]]$.

(ii) Let $\bar{u}$ be a positive solution of (14). Then we have $(\varphi(x,\bar{u})\Delta + f(x,\bar{u}))\bar{u} = 0$. This implies $\lambda_1[\varphi(x,\bar{u})\Delta + f(x,\bar{u})] = 0$. Then since $f \in F$ and $g \in G$, $\frac{f(x,u)}{\varphi(u)}$ is decreasing in u by Lemma 1. So by the last part of Lemma 3, we have $\lambda_1[\varphi(x,0)\Delta + f(x,0)] > \lambda_1[\varphi(x,\bar{u})\Delta + f(x,\bar{u})] = 0$, which is a contradiction.

Next we show that the above solution is unique. Suppose u_1 and u_2 are two positive solutions of (14). Let $\bar{u}$ be a maximal solution of (14). Then $\bar{u} \geq u_1$ and $\bar{u} \geq u_2$. If we can show $\bar{u} = u_1$ and $\bar{u} = u_2$, then $u_1 = u_2$. Thus it suffices to show that any other positive solution u_1

must coincide with $\bar{u}$. Since $\bar{u}$ and u_1 are solutions of (14), we have

$$\int_\Omega [-u_1 \Delta \bar{u} + \bar{u} \Delta u_1] = \int_\Omega \bar{u} u_1 \left[\frac{f(x,\bar{u})}{\varphi(x,\bar{u})} - \frac{f(x,u_1)}{\varphi(x,u_1)} \right] \tag{18}$$

The left integral of (18) is

$$\int_\Omega [-u_1 \Delta \bar{u} + \bar{u} \Delta u_1] = \int_{\partial\Omega} \left[-u_1 \frac{\partial \bar{u}}{\partial n} + \bar{u} \frac{\partial u_1}{\partial n} \right] = \int_{\partial\Omega} [u_1 \kappa \bar{u} - \bar{u} \kappa u_1] = 0.$$

Since $\frac{f(x,u)}{\varphi(x,u)}$ is decreasing in u by Lemma 1 , the right side of (18) is nonpositive. By the continuity of f and g, we must have $\bar{u} = u_1$.

The following lemma is a generalization of Lemma 4 [6]. Here the diffusion rate is nonlinear. According to Lemma 8, the equation (14) has a unique positive solution. We denote it by $u_{\varphi,f}$. Let u_{φ_n,f_n} be the unique positive solution of

$$\begin{cases} -\varphi_n(x,u)\Delta u = u f_n(x,u) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{19}$$

LEMMA 9. *Assume $f \in F$ and $\varphi \in G$.*

(i) $(\varphi, f) \mapsto u_{\varphi,f}$ is a continuous mapping of $G \times F \to C^{1,\alpha}(\Omega \times \mathbf{R}^+)$ for some $\alpha \in (0,1)$.

(ii) If $\frac{f_1}{\varphi_1} \geq \frac{f_2}{\varphi_2} \not\equiv \frac{f_1}{\varphi_1}$, for $x \in \Omega$, then either $u_{\varphi_1,f_1} > u_{\varphi_2,f_2}$ or $u_{\varphi_1,f_1} \equiv u_{\varphi_2,f_2} \equiv 0$.

Proof. (i) The argument is similar to that in Lemma 4 [6]. Here we just give a simple modification for the uniform boundedness of $\{u_{\varphi_n,f_n}\}$. Note that since $f_n \to f$ and $\varphi_n \to \varphi$, there exists $K > 0$ such that

$$\|u_{\varphi_n,f_n} f_n(x, u_{\varphi_n,f_n}) + P u_{\varphi_n,f_n}\|_\infty \leq K$$

Therefore, as in the proof of Lemma 2 with $h_n = u_{\varphi_n,f_n} f(x, u_{\varphi_n,f_n}) + P u_{\varphi_n,f_n}$, we have $\|u_{\varphi_n,f_n}\|_{W^{2,m}(\Omega)} \leq N$ by (9), and so there exists a subsequence $\{u_{\varphi_n,f_n}\}$ such that $u_{\varphi_n,f_n} \xrightarrow{w} \tilde{u}$ in $W^{2,m}(\Omega)$. Also we have by the Sobolev imbedding theorem, $\|u_{\varphi_n,f_n}\|_{C^{1,\alpha}(\Omega)} \leq M$ where M is a positive constant. Thus there exists a subsequence of $\{u_{\varphi_n,f_n}\}$, say

$\{u_{\varphi_n,f_n}\}$ again, such that $u_{\varphi_n,f_n} \to \tilde{u}$ in $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. As in the proof of Lemma 2, we conclude that $\tilde{u}$ is a solution. Then the positiveness of u_{φ_n,f_n} implies $\tilde{u} > 0$. So by the uniqueness of solution, $\tilde{u} = u_{\varphi,f}$. Therefore $u_{\varphi_n,f_n} \to u_{\varphi,f}$ in $C^{1,\alpha}(\bar{\Omega})$.

(ii) Suppose $\frac{f_1}{\varphi_1} > \frac{f_2}{\varphi_2}$. If we assume $\lambda_1(\varphi_1(x,0)\Delta + f_1(x,0)) \le 0$, then by Lemma 8, $u_{\varphi_1,f_1} \equiv u_{\varphi_2,f_2} \equiv 0$. So suppose $\lambda_1(\varphi_1(x,0)\Delta + f_1(x,0)) > 0$. Then $u_{\varphi_1,f_1} > 0$. Since $-\Delta u = u\frac{f_1(x,u)}{\varphi_1(x,u)} \ge u\frac{f_2(x,u)}{\varphi_2(x,u)}$, u_{φ_2,f_2} is a lower solution of (14). Also note that $u = c_0$ is an upper solution. Thus there exists a solution $\bar{u}$ of (14) such that $u_{\varphi_2,f_2} \le \bar{u} \le c_0$. By the uniqueness of positive solution of the equation (14), $\bar{u} = u_{\varphi_1,f_1}$. So we have $u_{\varphi_1,f_1} \ge u_{\varphi_2,f_2}$. Apply the strong maximal principle to get $u_{\varphi_1,f_1} > u_{\varphi_2,f_2}$.

3. Existence Theorem

In this section, we consider the system (1):

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u,v) \\ -\psi(x,v)\Delta v = vg(x,u,v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \end{cases}$$

For the predator-prey interaction, we make the following assumptions on the system.

(H1) $f, g \in C^1(\Omega \times \mathbf{R}^+ \times \mathbf{R}^+)$ satisfy

$$\begin{cases} f_u(x,u,v) < 0 \qquad f_v(x,u,v) < 0 \qquad \text{for } u, v > 0 \\ g_u(x,u,v) > 0 \qquad g_v(x,u,v) < 0 \qquad \text{for } u, v > 0 \end{cases}$$

Moreover, f_v, $g_u \ne 0$ and all partial derivatives are uniformly bounded on $\bar{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+$.

(H2) There exist positive constants C_1, C_2 such that

$$\begin{cases} f(x,u,0) < 0 \qquad \text{for } u > C_1 \\ g(x,0,v) < 0 \qquad \text{for } v > C_2 \\ g(x,C_1,C_2) < 0 \end{cases}$$

(H3) Let $\varphi(x,u)$, $\psi(x,v) \in C(\bar{\Omega} \times \mathbf{R}^+)$ and $\varphi(x,\cdot)$, $\psi(x,\cdot) \in G$.

The assumption (H1) describes how these species u, v interact with each other, while the assumption (H2) indicates that the model under consideration is logistic.

Let $\lambda_{1,\kappa}(A)$, $\lambda_{1,\sigma}(A)$ denote the first eigenvalues of an operator A under the boundary conditions $\frac{\partial\cdot}{\partial n} + \kappa\cdot = 0$ and $\frac{\partial\cdot}{\partial n} + \sigma\cdot = 0$, respectively.

By Lemma 8, if $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,0)) > 0$ where $f \in F$ and $\varphi \in G$, then the equation

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u,0) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

has a unique positive solution. Denote this positive solution by u_0.

Similarly, if $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,0,0)) > 0$ where $g \in F$ and $\psi \in G$, then

$$\begin{cases} -\psi(x,v)\Delta v = vg(x,0,v) \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \qquad \text{on } \partial\Omega \end{cases}$$

has a unique positive solution v_0.

According to the Lemma 9, we may define the operator $T : C(\bar{\Omega}) \to C(\Omega)$ as follows. For $v \in C(\Omega)$, by Lemma 8, $u = Tv$ is the unique positive solution to the equation

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u,v) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

if $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,v)) > 0$ where $f \in F$ and $\varphi \in G$.

REMARK. By Lemma 9, it is easy to see that T is a continuous operator and T is strictly monotone. In case that u is a prey for v, T is decreasing in v.

Replacing u by Tv in the other equation, we have

$$\begin{cases} -\psi(x,v)\Delta v = vg(x,Tv,v) \\ -\dfrac{\partial v}{\partial n} + \sigma v = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{20}$$

According to Lemma 2, we define the operator $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ by considering the equation

$$\begin{cases} -\psi(x,w)\Delta w + Pw = vg(x,Tv,v) + Pv \\ \dfrac{\partial w}{\partial n} + \sigma w = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{21}$$

We denote it by $w = Av = (-\psi(x,\cdot)\Delta \cdot + P\cdot)^{-1}[vg(x,Tv,v) + Pv]$ where P is constant. Note that the operator A has a fixed point v in the positive cone $\mathbf{K}$ if and only if (Tv, v) is a nonnegative solution of (1).

THEOREM 1. *Assume that (H1)-(H3) hold. If $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,u_0,0)) > 0$, then A has a positive fixed point in $\mathbf{K}$.*

Proof. To show this, we apply Lemma 4. Denote $A_\theta = \theta A$. If $A_\theta(v_\theta) = v_\theta$, $\theta \in [0,1]$, $v_\theta \in \mathbf{K}$, then we have

$$-\psi(x, \frac{v_\theta}{\theta})\Delta v_\theta = \theta v_\theta g(x, Tv_\theta, v_\theta) + (\theta - 1)Pv_\theta. \tag{22}$$

Suppose $v_\theta \not\equiv 0$. Let $v_\theta(x_0) = \max_{x\in\bar{\Omega}} v_\theta(x) > 0$ for some $x_0 \in \bar{\Omega}$. Then x_0 must be in Ω by the boundary condition of (21), and at x_0 the left side of (22) is nonnegative and $(\theta - 1)Pv_\theta(x_0) \leq 0$. Thus we must have $g(x_0, Tv_\theta(x_0), v_\theta(x_0)) \geq 0$. Suppose u, v are prey and predator, respectively. Since $g(x,Tv,v) \leq g(x,C_1,v)$ in $v \in \mathbf{K}$ where C_1 is *a priori* bound for u, we have $0 \leq g(x_0, Tv_\theta(x_0), v_\theta(x_0)) \leq g(x_0, C_1, v_\theta(x_0))$. Thus by the assumption (H2), $v_\theta(x_0) \leq C_2$. Thus there is an *a priori* bound for the positive fixed points of A_θ. Next observe $A'(0) = (-\psi(x,0)\Delta + P)^{-1}(g(x,u_0,0) + P)$ by Lemma 7, where $(-\psi(x,0)\Delta + P)^{-1}$ is a linear operator under boundary condition $\frac{\partial\cdot}{\partial n} + \sigma\cdot = 0$. Since $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,u_0,0)) > 0$, Lemma 6 implies $r(A'(0)) > 1$. Note that A is strongly positive compact operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$ by Lemma 2. Thus A has a positive fixed point v in $\mathbf{K}$ by Lemma 4.

THEOREM 2. *Suppose the conditions (H1)-(H3) hold.*

(a) If $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,0)) \leq 0$, then (1) has no positive solution.

(b) Suppose $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,0,0)) > 0$. *(therefore there exists a solution* $(0,v_0)$ *with* $v_0 > 0$.*) Then the system (1) has a positive solution if and only if* $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,v_0)) > 0$.

(c) Suppose $\lambda_{1,\sigma}(\psi(x,0)\Delta+g(x,0,0)) \leq 0$. *Then the system (1) has a positive solution if and only if* $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,0)) > 0$ *(thus there must be a solution* $(u_0,0)$ *with* $u_0 > 0$*) and* $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,u_0,0)) > 0$.

Proof. (a) Suppose $(\bar{u},\bar{v})$ is a positive solution of the system (1). Then $\bar{u} \not\equiv 0$ and $(\varphi(x,\bar{u})\Delta+f(x,\bar{u},\bar{v}))\bar{u} = 0$. As in the proof of Lemma 8 (ii), Lemma 1 and Lemma 3 implies $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,0)) > \lambda_{1,\kappa}(\varphi(x,\bar{u})\Delta + f(x,\bar{u},0)) > \lambda_{1,\kappa}(\varphi(x,\bar{u})\Delta + f(x,\bar{u},\bar{v})) = 0$, a contradiction. Thus the system has no positive solution.

(b) ($\Longleftarrow$) Since $\lambda_{1,\sigma}(\psi(x,0)\Delta+g(x,u_0,0)) > \lambda_{1,\sigma}(\psi(x,0)\Delta+g(x,0,0)) > 0$, A has a positive fixed point v in $\mathbf{K}$ by Theorem 1. Note that $u = Tv > 0$ because if $Tv \equiv 0$, then by uniqueness, $v_0 = v$. But $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,v_0)) > 0$ implies $Tv = Tv_0 > 0$, a contradiction. Therefore (Tv,v) is a positive solution of the system (1). ($\Longrightarrow$) Let $(\bar{u},\bar{v})$ be the positive solution of (1). Since $g(x,u,v) \geq g(x,0,v)$, we have $-\psi(x,v_0)\Delta v_0 \leq v_0 g(x,u,v_0)$. So v_0 is a lower solution of $-\psi(x,\bar{v})\Delta\bar{v} = \bar{v}g(x,u,\bar{v})$. Thus $v_0 \leq \bar{v}$. So we have $0 = \lambda_{1,\kappa}(\varphi(x,\bar{u})\Delta + f(x,u,\bar{v})) < \lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,\bar{v})) < \lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,v_0))$. Therefore $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,v_0)) > 0$.

(c) Again, since $\lambda_{1,\sigma}(\psi(x,0)\Delta+g(x,u_0,0)) > 0$, there exists a fixed point v of A by Theorem 1. Then $u = Tv > 0$ since if $Tv \equiv 0$, then $v = v_0$. Thus, by using Lemma 1 and Lemma 3, we have $0 = \lambda_{1,\sigma}(\psi(x,v)\Delta + g(x,0,v)) \leq \lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,0,0)) \leq 0$, a contradiction. For the necessity, let $\bar{u}$ be a positive solution with $\bar{v} = v$ of the system (1). If $\bar{v} = 0$, then $\bar{u} = u_0$. If $\bar{v} > 0$, then $u_0 > \bar{u}$ by the monotonicity of T. (See Remark.) Thus we have $u_0 \geq u > 0$. If $\lambda_{1,\kappa}(\varphi(x,0)\Delta + f(x,0,0)) \leq 0$, we have a contradiction by (a). If $\lambda_{1,\sigma}(\psi(x,0)\Delta + g(x,u_0,0)) \leq 0$, then $0 = \lambda_{1,\sigma}(\psi(x,v)\Delta + g(x,\bar{u},v)) < \lambda_{1,\sigma}(\psi(x,0) + g(x,\bar{u},0)) < \lambda_{1,\sigma}(\psi(x,0) + g(x,u_0,0)) \leq 0$, a contradiction again.

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