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HAUSDORFF DIMENSION OF SOME SPECIFIC PERTURBED CANTOR SET

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1. Introduction

We [1] investigated the Hausdorff dimension and the packing dimension of a certain perturbed Cantor set whose ratios are uniformly bounded. In this paper, we consider a specific perturbed Cantor set whose ratios are not necessarily uniformly bounded but satisfy some other conditions. In fact, in the hypothesis, only the condition of the uniform boundedness of ratios on the set is substituted by a "$*$-condition". We use energy theory related to Hausdorff dimension in this study while we [1] used Hausdorff density theorem to find the Hausdorff dimension of some perturbed Cantor set. In the end, we give an example which explains aforementioned facts.

2. Preliminaries

We recall the definition of perturbed Cantor set of [1]. Let $I_\phi = [0,1]$. We can obtain the left subinterval $I_{\sigma,1}$ and the right subinterval $I_{\sigma,2}$ of I_σ deleting middle open subinterval of I_σ inductively for each $\sigma \in \{1,2\}^n$, where $n = 0, 1, 2, \cdots$. Consider $E_n = \cup_{\sigma\in\{1,2\}^n} I_\sigma$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we fix $\mid I_{\sigma,1} \mid / \mid I_\sigma \mid = a_{n+1}$ and $\mid I_{\sigma,2} \mid / \mid I_\sigma \mid = b_{n+1}$ for all $\sigma \in \{1,2\}^n$, where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^{\infty} E_n$ a perturbed Cantor set.

We recall the s-dimensional Hausdorff measure of F :

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

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where $H^s_\delta(F) = \inf\{\sum_{n=1}^\infty |U_n|^s \colon \{U_n\}_{n=1}^\infty$ is a δ-cover of $F\}$, and the Hausdorff dimension of F :

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}$$
$$(= \inf\{s > 0 : H^s(F) = 0\})(\text{see}[2]).$$

We note that if $\{a_n\}$ and $\{b_n\}$ are given, then a perturbed Cantor set F is determined, *vice versa*. We are now ready to study the ratio geometry of the perturbed Cantor set.

3. Main results

In this section, F means a perturbed Cantor set determined by $\{a_n\}$ and $\{b_n\}$. We introduce $*$-condition which plays an important role in the energy theory related to Hausdorff dimension.

Let $s \in (0,1)$. F is said to satisfy $*$-condition for s, if for each $\delta > 0$ there exist positive $\epsilon < 1$ and N such that $\prod_{i=N}^k (a_i^s + b_i^s) \prod_{i=1}^k [\max(a_i, b_i)]^\delta < (1-\epsilon)^{k-N}$ for all $k \geq N$.

Before going into the main theorem, it is fruitful to know some properties of the ratios $\{a_n\}, \{b_n\}$ of a perturbed Cantor set F.

LEMMA 1. *Let $\{a_n\}, \{b_n\}$ be sequences in (0,1) and $d_n = 1 - (a_n + b_n) > 0$ for each n. Then*

(1) *there exists unique s_n such that $a_n^{s_n} + b_n^{s_n} = 1$ for each $n = 1, 2, \cdots$.*
(2) $0 \leq \underline{s} \leq \overline{s} \leq 1$, *where* $\underline{s} = \liminf_{n\to\infty} s_n$, $\overline{s} = \limsup_{n\to\infty} s_n$.
(3) *if* $\liminf_{n\to\infty} a_n > 0$ *and* $\liminf_{n\to\infty} b_n > 0$, *then* $\limsup_{n\to\infty} (a_n^{\overline{s}} + b_n^{\overline{s}}) = 1$.
(4) *if* $\liminf_{n\to\infty} \sum_{k=1}^n \log(a_k^s + b_k^s)$ *is finite for some s, then* $\underline{s} \leq s \leq \overline{s}$.
(5) *if* $\liminf_{n\to\infty} \sum_{k=1}^n \log(a_k^s + b_k^s)$ *is finite for some s, then there is some $B < \infty$ such that* $\liminf_{n\to\infty} \prod_{i=k}^n \log(a_i^s + b_i^s) < B$ *for any k.*
(6) *if* $\limsup_{n\to\infty}(a_n^s + b_n^s) \leq 1$ *for some s and* $\liminf_{n\to\infty} d_n > 0$, *then F satisfies the $*$-condition for s.*

Proof. (1) - (5) follow immediately from the definitions of lim inf and lim sup. For (6), we can find $d > 0$ such that $a_n + b_n \leq 1 - d$ for all

$n \geq N_1$ for some N_1 since $\liminf_{n\to\infty} d_n > 0$. Given $\delta > 0$, we choose $0 < \epsilon < 0$ such that $(1-d)^\delta < 1 - 2\epsilon$. For such $\epsilon > 0$, there is $N \geq N_1$ such that $a_n^s + b_n^s < 1 + \epsilon$ for all $n \geq N$. Therefore for all $k \geq N$, $\prod_{i=N}^k (a_i^s + b_i^s) \prod_{i=1}^k [\max(a_i, b_i)]^\delta < \prod_{i=N}^k (a_i^s + b_i^s) \prod_{i=1}^k (1-d)^\delta < \prod_{i=N}^k (1+\epsilon)(1-2\epsilon) < \prod_{i=N}^k (1+\epsilon)(\frac{1-\epsilon}{1+\epsilon}) < (1-\epsilon)^{k-N}$.

THEOREM 2. *If* $\liminf_{n\to\infty} \sum_{k=1}^n \log(a_k^s + b_k^s)$ *is finite for some* s *and* $\liminf_{n\to\infty} d_n > 0$ *and* F *satisfies the* $*$*-condition for* s*, then* $\dim_H(F) = s$.

Proof. We define a set function μ by

$$\mu(I_\sigma) = \liminf_{n\to\infty} \sum_{I \in I_\sigma \cap E_n} |I|^s$$

for each $\sigma \in \{1,2\}^k$, where $k = 1, 2, \cdots$. Clearly $\mu(I_\sigma) = \mu(I_{\sigma,1}) + \mu(I_{\sigma,2})$ for all σ. Then μ is extended to a mass distribution on F (see [2,Proposition1.7]).

For $x, y \in F$, we denote $x_\wedge y$ by the interval that both x and y belong to a common interval of E_n with greatest integer n. Since $\liminf_{n\to\infty} d_n > 0$, we find $d > 0$ such that $a_k + b_k \leq 1 - d$ for all $k \geq N_1$ for some N_1. Then for $\sigma \in \{1,2\}^k$ where $k \geq N_1$, the gap of $I_{\sigma,1}$ and $I_{\sigma,2}$ is not less than $d|I_\sigma|$.

Let $0 < t < s$. By slight abuse of notation, we write $I \in E_k$ to mean that the interval I is one of the basic intervals I_σ of E_k, where $\sigma \in \{1,2\}^k$.

If $x_\wedge y \in E_k$ and $k \geq N_1$, then for $\sigma \in \{1,2\}^k$,

$$\begin{aligned}
&\int\int_{x_\wedge y = I_\sigma} |x-y|^{-t} d\mu(x) d\mu(y) \\
\leq& 2d^{-t} |I_\sigma|^{-t} \mu(I_{\sigma,1}) \mu(I_{\sigma,2}) \\
=& 2d^{-t} |I_\sigma|^{-t} \liminf_{n\to\infty} \sum_{I \in I_{\sigma,1} \cap E_n} |I|^s \liminf_{n\to\infty} \sum_{I' \in I_{\sigma,2} \cap E_n} |I'|^s \\
\leq& 2d^{-t} |I_\sigma|^{-t} \liminf_{n\to\infty} \sum_{I \in I_{\sigma,1} \cap E_n, I' \in I_{\sigma,2} \cap E_n} |I|^s |I'|^s
\end{aligned}$$

$$=2d^{-t}\liminf_{n\to\infty}\prod_{i=k+2}^{n}(a_i^s+b_i^s)^2|I_\sigma|^{-t}|I_{\sigma,1}|^s|I'_{\sigma,2}|^s$$

$$\le 2d^{-t}\liminf_{n\to\infty}\prod_{i=k+2}^{n}(a_i^s+b_i^s)^2|I_\sigma|^{2s-t}.$$

Using Lemma 1 (5), for $\sigma\in\{1,2\}^k$ where $k\ge N_1$, we obtain

$$\int\int_{x\wedge y=I_\sigma}|x-y|^{-t}d\mu(x)d\mu(y)$$
$$\le 2d^{-t}B|I_\sigma|^{2s-t}\text{for some } B<\infty.$$

Since F satisfies the $*$-condition for s, there exist positive $\epsilon<1$ and $N\ge N_1$ such that $\prod_{i=N}^{k}(a_i^s+b_i^s)\prod_{i=1}^{k}[\max(a_i,b_i)]^{s-t}<(1-\epsilon)^{k-N}$ for all $k\ge N$. Therefore for all $k\ge N$, we have

$$\sum_{I\in E_k}|I|^{2s-t}=\sum_{I\in E_k}|I|^s|I|^{s-t}$$
$$\le\prod_{i=1}^{k}\{(a_i^s+b_i^s)[\max(a_i,b_i)]^{s-t}\}$$
$$<\alpha(1-\epsilon)^{k-N},$$

where $\alpha=\prod_{i=1}^{N-1}(a_i^s+b_i^s)$. Thus for $k\ge N$, we have

$$\sum_{k=N}^{\infty}\sum_{I\in E_k}\int\int_{x\wedge y=I}|x-y|^{-t}d\mu(x)d\mu(y)$$
$$\le 2d^{-t}B\sum_{k=N}^{\infty}\sum_{I\in E_k}|I|^{2s-t}$$
$$\le 2d^{-t}B\frac{\alpha}{\epsilon}.$$

Since $d_n>0$ for each n, $d^*=\min_{1\le k\le N}d_k$ exists. If $x\wedge y\in E_k$ and $k<N$, then we similarly obtain

$$\int\int_{x\wedge y=I_\sigma}|x-y|^{-t}d\mu(x)d\mu(y)$$
$$\le 2(d^*)^{-t}|I_\sigma|^{-t}\mu(I_{\sigma,1})\mu(I_{\sigma,2})$$
$$\le 2(d^*)^{-t}B|I_\sigma|^{2s-t}$$

for $\sigma \in \{1,2\}^k$. Thus

$$\begin{aligned}
&\sum_{k=0}^{N-1} \sum_{I \in E_k} \int\int_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \\
\leq & 2(d^*)^{-t} B \sum_{k=0}^{N-1} \sum_{I \in E_k} |I|^{2s-t} \\
\leq & 2(d^*)^{-t} B \sum_{k=0}^{N-1} 2^k \\
\leq & 2(d^*)^{-t} B 2^N .
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_F \int_F |x-y|^{-t} d\mu(x) d\mu(y) \\
=&\sum_{k=0}^{\infty} \sum_{I \in E_k} \int\int_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \\
=&\sum_{k=0}^{N-1} \sum_{I \in E_k} \int\int_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \\
&+\sum_{k=N}^{\infty} \sum_{I \in E_k} \int\int_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \\
\leq & 2(d^*)^{-t} B 2^N + 2d^{-t} B \frac{\alpha}{\epsilon} < \infty .
\end{aligned}$$

Since μ is a mass distribution on F, $\dim_H(F) \geq s$ (see [2,Theorem 4.13 (a)]). Now $\dim_H(F) \leq s$ follows easily from the fact that $\liminf_{n\to\infty} \sum_{k=1}^{n} \log(a_k^s + b_k^s) < \infty$ and $\liminf_{n\to\infty} d_n > 0$.

COROLLARY 3. *If* $\liminf_{n\to\infty} \sum_{k=1}^{n} \log(a_k^s + b_k^s)$ *is finite,* $\liminf_{n\to\infty} d_n > 0$ *and* $\limsup_{n\to\infty}(a_n^s + b_n^s) \leq 1$, *then* $\dim_H(F) = s$.

Proof. It follows immediately from Theorem 2 and Lemma 1 (6).

EXAMPLE 4. Consider a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $n_{k+1} - n_k > k + 1$ for all k. Let $0 < \epsilon < \frac{1}{4}$ and $b_{n_k} = 1 - \epsilon$ for all k. We also take a decreasing sequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $0 < a_{n_k} \le \frac{\epsilon}{2}$ with $\lim_{k\to\infty} a_{n_k} = 0$, and $\log(a_{n_k}^{\frac{1}{2}} + b_{n_k}^{\frac{1}{2}}) < 0$. Let $b_{n_k+i} = \frac{1}{4}, 1 \le i \le k$ for each k. For each k and $1 \le i \le k$, we find a_{n_k+i} such that $\log\{(a_{n_k+i})^{\frac{1}{2}} + (b_{n_k+i})^{\frac{1}{2}}\} = \frac{\delta_k}{k}$, where $\delta_k = -\log(a_{n_k}^{\frac{1}{2}} + b_{n_k}^{\frac{1}{2}})$. We put $a_n = b_n = \frac{1}{4}$ if $n \ne n_k, n_k + 1, \cdots, n_k + k$, where $k = 1, 2, \cdots$. Then $\liminf_{n\to\infty} a_n = 0$, and $\liminf_{n\to\infty} d_n > 0$. In fact, $d_n \ge \min(\frac{\epsilon}{2}, 1 - \{\frac{1}{4} + (\frac{4\sqrt{3}-3}{6})^2\}) > 0$ for all n. Clearly $\limsup_{n\to\infty}(a_n^{\frac{1}{2}} + b_n^{\frac{1}{2}}) = 1$ and $\liminf_{n\to\infty} \sum_{k=1}^{n} \log(a_k^{\frac{1}{2}} + b_k^{\frac{1}{2}}) = \log(1-\epsilon)^{\frac{1}{2}}$. Using Corollary 3, we obtain $\dim_H(F) = \frac{1}{2}$.

References

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