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FOUNDATIONS OF THE KKM THEORY VIA COINCIDENCES OF COMPOSITES OF UPPER SEMICONTINUOUS MAPS

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1. Introduction

In the KKM theory, there exist mutually equivalent fundamental theorems from which most of the other important results in the theory can be deduced. We give such fundamental theorems related to composites of upper semicontinuous (u.s.c.) multifunctions in a very large class.

An u.s.c. multifunction with nonempty compact convex values will be called a *Kakutani map*. Recently, Simons [**Si1**] and Lassonde [**L2**] extended the well-known fixed point theorems due to Kakutani [**Kk**] and Himmelberg [**Hi**] to multifunctions factorizable by Kakutani maps through convex sets in topological vector spaces. Such multifunctions arise in a natural way in minimax and coincidence theories. For the literature, see [**L2**], [**GL2**], [**Gr1-3**]. Later Lassonde [**L4**] obtained the same results for a larger class of multifunctions.

On the other hand, Ben-El-Mechaiekh [**Bn1**] obtained an elementary proof of a fairly general fixed point theorem for composites of Kakutani maps defined on a class of general extension spaces containing locally convex and some not necessarily locally convex topological vector spaces. He also deduced some general coincidence theorems for composites of multifunctions. The aim in [**L2**], [**Bn1**] lies to give elementary approach to the convex-valued multifunctions not using homological methods. Recently, Ben-El-Mechaiekh and Deguire [**BD1-3**] generalized the main results of [**Bn1**] to a very large class of "admissible" u.s.c. maps with non-convex values.

An u.s.c. multifunction with compact acyclic values will be called an *acyclic map*. In [**P5**], the results in [**Si1**] were generalized to acyclic

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maps. Later the present author [**P6**] obtained a general Fan-Browder type coincidence theorem related to acyclic maps, and gave its equivalent formulations to fundamental theorems in the KKM theory. Moreover, a number of its applications were given to the KKM theory and fixed point theory. Further applications to acyclic maps were also given in the author's previous works [**P7-9**].

The purpose in this paper is, first, to establish some coincidence theorems for composites of multifunctions including a class of very general u.s.c. maps. Consequently, we obtain generalizations of main results of [**L2,4**], [**Bn1**], [**BD1-3**], [**P6**] to a class of maps which properly includes that of multifunctions factorizable by Kakutani or acyclic maps. Secondly, we show that fundamental theorems in the KKM theory can be obtained in far-reaching generalized forms related to such class of maps. Those are the KKM theorem, the matching theorem, the Fan-Browder fixed point theorem, the Ky Fan minimax inequality, analytic alternatives, geometric properties of convex sets, and others.

Our new results extend, improve, and unify main theorems in more than one hundred published works.

2. Preliminaries

We mainly follow [**Br**], [**P6**].

A *multifunction* (or *map*) $F : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $Fx \subset Y$ for $x \in X$ and the *fibers* $F^-y = \{x \in X : y \in Fx\}$ for $y \in Y$. As usual, the set $\{(x, y) : y \in Fx\}$ is called either the *graph* of F, or, simply, F. Therefore $(x, y) \in F$ if and only if $y \in Fx$.

For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. For any $B \subset Y$, the *lower inverse* and *upper inverse* of B under F are defined by

$$F^-(B) = \{x \in X : Fx \cap B \neq \emptyset\} \text{ and } F^+(B) = \{x \in X : Fx \subset B\},$$

resp. The (*lower*) *inverse* of $F : X \to 2^Y$ is the multifunction $F^- : Y \to 2^X$ defined by $x \in F^-y$ if and only if $y \in Fx$. Given two multifunctions $F : X \to 2^Y$ and $G : Y \to 2^Z$, the *composite* $GF : X \to 2^Z$ is defined by $(GF)x = G(Fx)$ for $x \in X$.

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^-(B)$

is closed in X; and *compact* if $F(X)$ is contained in a compact subset of Y. Int and $^{-}$ denote the interior and closure, resp. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A topological vector space is abbreviated as t.v.s.

For a set D, let $\langle D \rangle$ denote the set of nonempty finite subsets of D.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any $N \in \langle D \rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. Such convex hull will be called a *polytope*. A subset A of (X, D) is said to be *D-convex* if, for any $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$, where co denotes the convex hull. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [**L1**]. Note that for a convex space (X, D), X itself is not necessarily convex. For example, let X be any space containing an n-simplex Δ_n as a subspace and D the set of vertices of Δ_n. Then (X, D) is a convex space, and X may not be convex, but D-convex.

For a convex space (X, D), a multifunction $G : D \to 2^X$ is called a *KKM map* if $\operatorname{co} N \subset G(N)$ for each $N \in \langle D \rangle$. The *KKM theory* is the study of KKM maps and their applications. For the literature, see [**Au**], [**AE**], [**Gr2,3**], [**P6**], [**Z**].

Given a class $\mathbb{L}$ of multifunctions, $\mathbb{L}(X, Y)$ denotes the set of multifunctions $T : X \to 2^Y$ belonging to $\mathbb{L}$, and $\mathbb{L}_c$ the set of finite composites of multifunctions in $\mathbb{L}$.

Let (X, D) be a convex space and Y a topological space. Define

$T \in \Phi(X, Y) \iff T^- y$ is D-convex for each $y \in Y$ and $\{\operatorname{Int} Tx : x \in D\}$ covers Y.

$T \in \mathbb{M}(X, Y) \iff T^-|_K$ has a continuous selection $s : K \to X$ for every nonempty compact subset K of Y such that $s(K) \subset P$ for some polytope P of (X, D).

For a topological space X, define

$T \in \mathbb{K}(X, Y) \iff T$ is a Kakutani map; that is, Y is a convex space and T is u.s.c. with nonempty compact convex values.

$T \in \mathbb{V}(X, Y) \iff T$ is an acyclic map; that is, T is u.s.c. with compact acyclic values.

We now introduce an abstract class $\mathfrak{A}$ of multifunctions motivated by Ben-El-Mechaiekh and Deguire [**BD2,3**]:

A class $\mathfrak{A}$ of multifunctions is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Note that $\mathbb{C}$, $\mathbb{K}$, and $\mathbb{V}$ are examples of $\mathfrak{A}$ by a Lefschetz type fixed point theorem for composites of acyclic maps due to Górniewicz and Granas [**GG1,2**]. See also [**Si1**], [**P5**]. Moreover, the class of approachable maps in topological vector spaces is an example of $\mathfrak{A}$ [**BD3**]. For other examples of related classes, see [**BD1-3**], [**L4**].

Further, we define the following:

$T \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Tx$ for each $x \in K$.

$T \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ as above.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. The class $\mathbb{K}_c^\sigma$ is due to Lassonde [**L4**] and $\mathbb{V}_c^\sigma$ to Park, Singh, and Watson [**PSW**]. Note that $\mathbb{K}_c^\sigma$ includes the classes $\Phi^- = \{F^- : F \in \Phi\}$, $\mathbb{M}^-$, and $\mathbb{T}$ in [**L4**].

3. Coincidences of compact composites of maps

We begin with the following coincidence theorem:

THEOREM 1. *Let (X, D) be a convex space, Y a topological space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $G \in \mathbb{M}_c(X, Y)$. If F is compact, then F and G have a coincidence point; that is, there exists an $x_0 \in X$ such that $Fx_0 \cap Gx_0 \neq \emptyset$.*

Proof. Since F is compact, there exists a compact set K such that $F(X) \subset K \subset Y$. Since $G \in \mathbb{M}_c(X, Y) = \mathbb{M}(X, Y)$, $G^-|_K$ has a continuous selection $s : K \to P \subset X$, where P is a polytope of (X, D). Then $(sF)|_P \in \mathfrak{A}_c^\kappa(P, P)$ has a fixed point $x_0 \in P \subset X$. Since $x_0 \in (sF)x_0$, $Fx_0 \cap Gx_0 \supset Fx_0 \cap s^-(x_0) \neq \emptyset$. This completes our proof.

REMARK. In fact, $F \in \mathfrak{A}_c^\kappa(X, Y)$ can be replaced by any $F : X \to 2^Y$ such that $F \in \mathfrak{A}_c^\kappa(P, Y)$ for any polytope P in (X, D). Some equivalent or particular forms of Theorem 1 for $\mathbb{V}$ replacing $\mathfrak{A}_c^\kappa$ were adopted by Ben-El-Mechaiekh *et al.* [**BDG2,3**], Ha [**Ha1**], Granas and Liu [**GL2**], Komiya [**Ko**], Ben-El-Mechaiekh [**Bn1**], and Park [**P6**] as the starting point of some of their own studies.

We obtain the following extended version of a coincidence theorem due to Granas and Liu [**GL2**].

THEOREM 2. *Let (X, D) be a convex space, Y a Hausdorff space, and $F, G : X \to 2^Y$ multifunctions satisfying*

(2.1) *$F \in \mathfrak{A}_c^\kappa(X, Y)$ is compact;*
(2.2) *for each $y \in F(X)$, $G^- y$ is D-convex; and*
(2.3) *$\{\operatorname{Int} Gx : x \in D\}$ covers $\overline{F(X)}$.*

Then F and G have a coincidence point $x_0 \in X$; that is, $Fx_0 \cap Gx_0 \neq \emptyset$.

Proof. Since $\overline{F(X)}$ is compact and included in $\bigcup\{\operatorname{Int} Gx : x \in D\}$, there exists an $N = \{x_1, x_2, \ldots, x_n\} \in \langle D \rangle$ such that $\overline{F(X)} \subset \bigcup\{\operatorname{Int} Gx : x \in N\}$. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to this cover, and $P = \operatorname{co} N \subset X$. Define $f : \overline{F(X)} \to P$ by

$$fy = \sum_{i=1}^{n} \lambda_i(y) x_i = \sum_{i \in N_y} \lambda_i(y) x_i$$

for $y \in \overline{F(X)} \subset Y$, where

$$i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in \operatorname{Int} Gx_i.$$

Then $x_i \in G^- y$ for each $i \in N_y$. Clearly f is continuous and, by (2.2), we have $fy \in \operatorname{co}\{x_i : i \in N_y\} \subset G^- y$ for each $y \in F(X)$. Since $P \subset X$ and $F|_P \in \mathfrak{A}_c^\kappa(P, F(X))$, $f(F|_P) : P \to 2^P$ has a fixed point $x_0 \in P \subset X$. Since $x_0 \in (fF)x_0$ and $f^- x_0 \subset Gx_0$, we have $Fx_0 \cap Gx_0 \neq \emptyset$. This completes our proof.

REMARKS. 1. As in Theorem 1, (2.1) can be replaced by

(2.1)′ $F \in \mathfrak{A}_c^\kappa(P, Y)$ for any polytope P in (X, D)

without affecting the conclusion of Theorem 2.

2. If we assume

(2.2)′ for each $y \in \overline{F(X)}$, $G^- y$ is D-convex

instead of (2.2), then Theorem 2 follows from Theorem 1. In this case, the proof works for that of $\Phi \subset \mathbb{M}$.

Particular Forms. The origin of Theorem 2 goes back to Browder [**Bw1-3**] for $X = D = Y$ and $F = 1_X$. For $X = D$ and $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, Theorem 2 reduces to the main result of Granas and Liu [**GL2**]. For $X = D$ and $\mathbb{K}_c^\sigma$ instead of $\mathfrak{A}_c^\kappa$, Theorem 2 reduces to Lassonde [**L4**, Théorème 4]. Moreover, [**PSW**, Theorem 1] is a particular case of Theorem 2 for $X = D$ and $\mathbb{V}_c^\sigma$. For other cases, see the particular forms of Theorem 5.

From Theorem 2, we have the following:

Theorem 3. *Let X and C be nonempty convex subsets of a locally convex Hausdorff t.v.s. E, and $F \in \mathfrak{A}_c(X, X + C)$ a compact multifunction. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*
(ii) *X is compact and C is closed.*
(iii) *$C = \{0\}$.*

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of the origin 0 in E, and Y a compact set such that $F(X) \subset Y \subset X + C$. Define $G : X \to 2^Y$ by $Gx = (x + C + V) \cap Y$ for $x \in X$. Then each Gx is open in Y and $G^- y = (y - C - V) \cap X$ is convex for each $y \in Y$. Moreover, since $Y \subset X + C$, for every $y \in Y$, there exists an $x \in X$ such that $y \in x + C + V$; that is, $\{Gx : x \in X\}$ covers Y. Therefore, by Theorem 2, there exist $x_V \in X$ and $y_V \in Y$ such that $y_V \in Fx_V \cap Gx_V$; that is, $y_V - x_V \in C + V$. In other words, we obtain the assertion:
$(*)$ for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion. Now we consider Cases (i)–(iii).

Case (i). Since X is closed so is $(F - i)(X)$. Since C is compact and E is regular, $(*)$ implies $(F - i)(X) \cap C \neq \emptyset$; that is, there exists an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Case (ii). Since $(F - i)(X)$ is compact and C is closed, the same conclusion follows as in Case (i).

Case (iii). Since F is u.s.c., for each neighborhood V of 0 in E, there exist $x_V,\ y_V \in X$ such that $y_V \in Fx_V$ and $y_V - x_V \in V$. Since $F(X)$ is relatively compact, we may assume that y_V converges to some

$\hat{x}$. Then x_V also converges to $\hat{x}$. Since the graph of F is closed in $X \times \overline{F(X)}$, we have $\hat{x} \in F\hat{x}$.

This completes our proof.

PARTICULAR FORMS. For $\mathbb{V}$ instead of $\mathfrak{A}_c$, Theorem 3 is due to Park [**P6**, Theorem 7], which extends Lassonde [**L1**, Theorem 1.6 and Corollary 1.18] for $\mathbb{K}$. For $\mathbb{V}_c$ instead of $\mathfrak{A}_c$, Theorem 3 includes Park, Singh, and Watson [**PSW**, Theorem 2].

THEOREM 4. *Let X be a nonempty convex subset of a locally convex Hausdorff t.v.s. E, and $F \in \mathfrak{A}_c^\sigma(X, X)$. If F is compact, then F has a fixed point.*

Proof. Let $M = \operatorname{co} \overline{F(X)}$. Then $M \subset X$ since $\overline{F(X)} \subset X$ and X is convex. Also M is σ-compact [**L4**, Proposition 1(3)]. Since $F \in \mathfrak{A}_c^\sigma(X, M)$, there exists a $\Gamma \in \mathfrak{A}_c(M, M)$ such that $\Gamma x \subset Fx$ for all $x \in M$. Since Γ is compact and M is a nonempty convex subset of E, by Theorem 3(iii), Γ has a fixed point $x_0 \in M$; that is, $x_0 \in \Gamma x_0 \subset F x_0$. This completes our proof.

PARTICULAR FORMS. 1. Theorem 4 includes Ben-El-Mechaiekh and Deguire [**BD2**, Corollaries 3.4 and 3.6], Ben-El-Mechaiekh [**Bn3**, Theorem 2.4], Lassonde [**L2**, Theorem 4], [**L4**, Théorème 5], Ben-El-Mechaiekh *et al.* [**BDG2**], Simons [**Si1**], Powers [Pw, Theorem 7.1], and Himmelberg [**Hi**].

2. Himmelberg's theorem; that is, Theorem 4 for $\mathbb{K}$, generalizes earlier works of Schauder [**S**], Mazur [**M**], Bohnenblust and Karlin [**BK**], Hukuhara [**Hu**], and Singbal [**Sn**]. Further if X itself is compact, then X is an *lc* space, and Theorem 4 for $\mathbb{V}$ follows from Begle [**Be**]. For this case, it generalizes well-known results of Brouwer [**B**], Schauder [**S**], Tychonoff [**Ty**], Kakutani [**Kk**], Fan [**F1**], and Glicksberg [**G**]. For the literature, see Dugundji and Granas [**DG2**] or Park [**P6**].

REMARKS. 1. If $F \in \mathfrak{A}_c(X, Y)$, then $F \in \mathfrak{A}_c^\sigma(X, Y)$. Therefore, Theorem 3(iii) and Theorem 4 are equivalent.

2. If E is real and F is a convex-valued compact multifunction, then the upper semicontinuity, upper demicontinuity, and upper hemicontinuity of F are all the same. See Shih and Tan [**ST5**]. Therefore, in this case, Theorem 4 for $\mathbb{K}$ can be applied to an upper hemicontinuous multifunction F.

4. The KKM theorems related to non-compact composites of maps

In this section, we show that Theorem 2 is equivalent to far-reaching generalizations of the Fan-Browder fixed point theorem, the Ky Fan matching theorem for open covers, the Knaster-Kuratowski-Mazurkiewicz theorem, and a whole intersection theorem. Those are a part of the very useful fundamental theorems in the KKM theory.

We begin with the following non-compact version of Theorem 2, which generalizes the main result of our previous work [**P6**].

THEOREM 5. *Let (X, D) be a convex space, Y a Hausdorff space, $S : D \to 2^Y$, $T : X \to 2^Y$ multifunctions, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(5.1) *for each $x \in D$, $Sx \subset Tx$ and Sx is compactly open;*
(5.2) *for each $y \in F(X)$, $T^- y$ is D-convex;*
(5.3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(5.4) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that $F(L_N) \backslash K \subset S(L_N \cap D)$.*

Then F and T have a coincidence point.

Proof. Since $\overline{F(X)} \cap K$ is compact, by (5.3), there exists an $N \in \langle D \rangle$ such that $\overline{F(X)} \cap K \subset S(N)$. Let L_N be the set in (5.4). Since L_N is compact and $F \in \mathfrak{A}_c^\kappa(X, Y)$, there exists a $\Gamma \in \mathfrak{A}_c(L_N, Y)$ such that $\Gamma x \subset Fx$ for each $x \in L_N$.

We claim that $\Gamma(L_N) \subset S(L_N \cap D)$. Note that

$$\Gamma(L_N) \cap K \subset F(X) \cap K \subset S(N) \subset S(L_N \cap D).$$

On the other hand, $\Gamma(L_N) \backslash K \subset F(L_N) \backslash K \subset S(L_N \cap D)$ by (5.4). Hence, we have $\Gamma(L_N) \subset S(L_N \cap D)$.

Note that $\Gamma(L_N)$ is compact since it is the image of the compact set L_N under the composite Γ of compact-valued u.s.c. multifunctions.

Now we use Theorem 2 with $(\Gamma|_{L_N}, T|_{L_N}, L_N, L_N \cap D)$ replacing (F, G, X, D). Note that $T|_{L_N}$ has D-convex fibers $T^- y \cap L_N$ for each $y \in \Gamma(L_N)$ by (5.2), and $S(L_N \cap D)$ covers $\Gamma(L_N)$; that is, $\{\operatorname{Int} Tx : x \in L_N \cap D\}$ covers $\overline{\Gamma(L_N)} = \Gamma(L_N)$. Hence, all of the requirements are satisfied, and thus, $\Gamma|_{L_N}$ and $T|_{L_N}$ have a coincidence point $x_0 \in L_N$. This completes our proof.

REMARKS. 1. Theorem 5 is actually equivalent to Theorem 2. In fact, we derived Theorem 5 from Theorem 2. Note that the converse is clear: By putting $Y = K$ and $T = G$, Theorem 5 reduces to Theorem 2.

2. The coercivity condition (5.4) is motivated by Chang [**C**].

3. Note that, if F is single-valued, the Hausdorffness assumption on Y is not necessary. See the argument in [**P6**]

PARTICULAR FORMS. 1. The origin of Theorem 5 is the same as Theorem 2: Browder [**Bw1-3**] for $X = D = Y = K$ and $F = 1_X$. Note that numerous applications of the Fan-Browder fixed point theorem have appeared in various fields as fixed point theory, minimax theory, variational inequalities, and so on. See [**P2**].

2. For $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, Theorem 5 reduces to Park [**P6**, Theorem 1], which includes earlier works of Browder [**Bw1-3**], Tarafdar [**Tr1-4**], Tarafdar and Husain [**TH**], Ben-El-Mechaiekh *et al.* [**BDG1,2**], Yannelis and Prabhakar [**YP**], Lassonde [**L1,2**], Ko and Tan [**KT**], Simons [**Si1,2**], Takahashi [**Tk2**], Komiya [**Ko**], Mehta [**Me**], Mehta and Tarafdar [**MT**], Sessa [**Ss**], Jiang [**J1-3**], McLinden [**Mc**], Granas and Liu [**GL1,2**], Park [**P1,2,4**], and Chang [**C**].

Among the numerous applications of Theorem 5, we give only an abstract variational inequality:

COROLLARY 1. *Let (X, D) be a Hausdorff convex space, $p : X \times X \to (-\infty, \infty]$, $q : D \times X \to (-\infty, \infty]$, $h : X \to [-\infty, \infty]$ with $h \not\equiv \infty$, $F \in \mathfrak{A}_c^\kappa(X, X)$, and K a nonempty compact subset of X. Suppose that*

(1) *$q(x, y) \le p(x, y)$ for $(x, y) \in D \times X$, and*
$p(x, y) + h(y) \le h(x)$ for $x \in X$ and $y \in Fx$;

(2) *for each $x \in D$, $\{y \in X : q(x, y) + h(y) > h(x)\}$ is compactly open;*

(3) *for each $y \in F(X)$, $\{x \in X : p(x, y) + h(y) > h(x)\}$ is D-convex; and*

(4) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that*

$$F(L_N) \backslash K \subset \bigcup_{x \in L_N \cap D} \{y \in X : q(x, y) + h(y) > h(x)\}.$$

Then there exists a solution $y_0 \in \overline{F(X)} \cap K$ *of the variational inequality:*

$$q(x, y_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in D.$$

Moreover, the set of all solutions y_0 *is a compact subset of* $\overline{F(X)} \cap K$.

Proof. Define multifunctions $S : D \to 2^X$ and $T : X \to 2^X$ by

$$Sx = \{y \in X : q(x, y) + h(y) > h(x)\} \quad \text{for} \quad x \in D;$$

and

$$Tx = \{y \in X : p(x, y) + h(y) > h(x)\} \quad \text{for} \quad x \in X.$$

Suppose that there exists a $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \notin S(D)$. Then the conclusion follows. Therefore we may assume that $\overline{F(X)} \cap K \subset S(D)$. Then all of the requirements of Theorem 5 are satisfied. Hence, there exists an $x_0 \in X$ such that $Fx_0 \cap Tx_0 \neq \emptyset$. Let $y_0 \in Fx_0 \cap Tx_0$. Then $y_0 \in Fx_0$ and

$$p(x_0, y_0) + h(y_0) > h(x_0),$$

which contradicts (1). Moreover, the set of all solutions y_0 is the intersection

$$\bigcap_{x \in D} \{y \in \overline{F(X)} \cap K : q(x, y) + h(y) \leq h(x)\}$$

of compactly closed subsets of the compact set $\overline{F(X)} \cap K$. This completes our proof.

REMARK. If $X = K$ itself is compact, then $y_0 \in Fx_0$ for some $x_0 \in X$. Even for $F = 1_X$, Corollary 1 is a basis of existence theorems of many results concerning variational inequalities. For the literature, see [**Gr3**], [**P6**], [**Gw**].

PARTICULAR FORMS. For $F = 1_X$, there have appeared a lot of particular forms of Corollary 1. See Brézis, Nirenberg, and Stampacchia [**BNS**], Juberg and Karamardian [**JK**], Mosco [**Mo**], Allen [**Al**], Takahashi [**Tk1**], Gwinner [**Gw**], Lassonde [**L1**], Park [**P6**], and Ben-El-Mechaiekh [**Bn2**].

From Theorem 5, we obtain the following Ky Fan type matching theorem for open covers:

THEOREM 6. *Let (X, D) be a convex space, Y a Hausdorff space, $S : D \to 2^Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(6.1) *for each $x \in D$, Sx is compactly open;*
(6.2) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(6.3) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that $F(L_N) \setminus K \subset S(L_N \cap D)$.*

Then there exists an $M \in \langle D \rangle$ such that $F(\operatorname{co} M) \cap \bigcap \{Sx : x \in M\} \neq \emptyset$.

Proof. For each $y \in Y$, put $T^- y = \operatorname{co} S^- y$ which is the minimal D-convex set containing $S^- y$. This defines a multifunction $T : X \to 2^Y$. Then all of the requirements of Theorem 5 are satisfied. In fact, for each $x \in D$, $x \in S^- y$ for some $y \in Y$ implies $x \in T^- y$; that is, $Sx \subset Tx$. This and (6.1) imply (5.1). Moreover, $T^- y$, which may be empty, is D-convex for each $y \in F(X)$. This shows (5.2). Since (6.2) and (6.3) are same as (5.3) and (5.4), resp., by Theorem 5, T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$. For $y \in Tx_0 \cap Fx_0$, we have $x_0 \in T^- y = \operatorname{co} S^- y$, and hence, there exists a finite set $M = \{x_1, x_2, \ldots, x_n\}$ in $S^- y \subset D$ such that $x_0 \in \operatorname{co} \{x_1, x_2, \ldots, x_n\}$. Since $x_i \in S^- y$ implies $y \in Sx_i$ for all i, $1 \leq i \leq n$, we have $y \in Fx_0 \cap \bigcap_{i=1}^n Sx_i$. This completes our proof.

PARTICULAR FORMS. 1. The origin of Theorem 6 goes back to Fan [**F7,9**] for $X = Y$ and $F = 1_X$.

2. For $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, Theorem 6 reduces to Park [**P6**, Theorem 2], which includes earlier results in [**P1,4**].

Theorem 6 can be stated in its contrapositive form and in terms of the complement Gx of Sx in Y. Then we obtain the following KKM theorem:

THEOREM 7. *Let (X, D) be a convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \to 2^Y$ be a multifunction such that*

(7.1) *for each $x \in D$, Gx is compactly closed in Y;*
(7.2) *for any $N \in \langle D \rangle$, $F(\operatorname{co} N) \subset G(N)$; and*
(7.3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap \{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold. Then $\overline{F(X)} \cap K \subset S(D)$ where $Sx = Y \backslash Gx$ for $x \in D$. Note that (7.1) and (7.3) imply (6.1) and (6.3), resp. Therefore, by Theorem 6, there exists an $M \in \langle D \rangle$ such that $F(\mathrm{co}\, M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$; that is, $F(\mathrm{co}\, M) \not\subset G(M)$. This contradicts (7.2).

REMARK. Condition (7.2) is equivalent to $\mathrm{co}\, N \subset F^+G(N)$; that is, the multifunction F^+G is a KKM-map. A KKM type theorem for this case different from Theorem 7 can be found in [**P6**, Theorem 4].

PARTICULAR FORMS. 1. The origin of Theorem 7 goes back to Sperner [**S**] and Knaster, Kuratowski, and Mazurkiewicz [**KKM**] for $X = Y = K = \Delta_n$ an n-simplex, D its set of vertices, and $F = 1_X$.

2. As we noted in [**P6**], a particular form [**P6**, Theorem 3] of Theorem 7 for $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$ includes earlier works of Fan [**F2,7,9**], Lassonde [**L1**], Chang [**C**], and Park [**P1,4**]. Moreover, in [**P9**], we showed that [**P6**, Theorem 3] extends a number of KKM type theorems recently due to Sehgal, Singh, and Whitfield [**SSW**], Lassonde [**La3**], Shioji [**So**], Liu [**Lu**], Chang and Zhang [**CZ**], and Guillerme [**Gu**].

From Theorem 7, we deduce two KKM type results of fundamental importance.

Recall that a family of sets is said to have the *finite intersection property* if the intersection of each finite subfamily is not empty.

COROLLARY 2. *Let* (X, D) *be a convex space,* Y *a Hausdorff space,* $F \in \mathfrak{A}_c^\kappa(X, Y)$, *and* $H : D \to 2^Y$ *such that, for any* $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset H(N)$. *Then the family* $\{\overline{Hx} : x \in D\}$ *has the finite intersection property*

Proof. Let $M \in \langle X \rangle$ and $L = \mathrm{co}\, M$. Let $\Gamma \in \mathfrak{A}_c(L, Y)$ such that $\Gamma x \subset Fx$ for all $x \in L$, $K = \Gamma(L)$, and $Gx = \overline{Hx}$ for $x \in M$. We use Theorem 7 with Γ replacing F. Conditions (7.1) and (7.2) hold clearly. Moreover, since $\Gamma(L_N) = \Gamma(L) \subset K$ for any $N \in \langle M \rangle$, (7.3) is satisfied automatically. Therefore, by Theorem 7, we have $\bigcap\{\overline{Hx} : x \in M\} \neq \emptyset$.

PARTICULAR FORMS. 1. The origin of Corollary 2 is due to Fan [**F2**] for a t.v.s. $E = X = Y$ and $F = 1_E$ and to Dugundji and Granas [**DG1**].

2. For $\mathbb{K}_c$ instead of $\mathfrak{A}_c^\kappa$, $X = D$, and a convex space Y, Corollary 2 reduces to Lassonde [**L2**, Theorem 2].

From Theorem 7, we can obtain the finite intersection property for an open-valued KKM map. We need the following due to Shih [**Sh**, Theorem 1]:

LEMMA. *Let X be a Hausdorff convex space and $M \in \langle X \rangle$. If $G : M \to 2^X$ is a KKM map with open values, then there is a KKM map $H : M \to 2^X$ with closed values such that $Hx \subset Gx$ for $x \in M$.*

COROLLARY 3. *Let (X, D) be a convex space, Y a Hausdorff space, $F : X \to 2^Y$ a compact-valued u.s.c. multifunction, and $G : D \to 2^Y$ such that*

(1) *for each $x \in D$, Gx is compactly open; and*
(2) *for any $N \in \langle D \rangle$, $\mathrm{co}\, N \subset F^+G(N)$.*

Then $\{Gx : x \in D\}$ has the finite intersection property.

Proof. Let $M \in \langle D \rangle$, $X_1 = \mathrm{co}\, M$, $F_1 = F|_{X_1}$, $G_1 = G|_M$, and $Y_1 = F_1(X_1)$. Then Y_1 is compact and we have

(1) for each $x \in M$, $(F_1^+ G_1)x$ is relatively open in X_1; and
(2) for each $N \in \langle M \rangle$, $\mathrm{co}\, N \subset F_1^+ G(N)$.

Therefore, $F_1^+ G_1 : M \to 2^{X_1}$ is a KKM map with open values. Then, by Lemma, there exists a KKM map $H : M \to 2^{X_1}$ with closed values such that $Hx \subset (F_1^+ G_1)x$ for each $x \in M$. So by Corollary 2 with $F = 1_{X_1}$,

$$\bigcap_{x \in M} (F_1^+ G_1)x \supset \bigcap_{x \in M} Hx \neq \emptyset;$$

that is, $\bigcap\{Gx : x \in M\} = \bigcap\{G_1 x : x \in M\} \neq \emptyset$. This completes our proof.

PARTICULAR FORMS. 1. The origin of Corollary 3 is due to Kim [**Ki1**] for $X = Y$ and $F = 1_X$.

2. Other forms of Corollary 3 can be seen in Kim [**Ki2**], Park [**P3,9**], and Lassonde [**L3**].

From Theorem 7, we have another whole intersection property as follows:

THEOREM 8. *Let (X, D) be a convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \to 2^Y$ be a multifunction satisfying (7.1) and (7.3). Suppose that there exists a multifunction $H : X \to 2^Y$ satisfying*

(8.1) *for each $x \in D$, $Hx \subset Gx$;*
(8.2) *for each $x \in X$, $Fx \subset Hx$; and*
(8.3) *for each $y \in F(X)$, $X \backslash H^- y$ is D-convex; that is, H has D-convex cofibers on $F(X)$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof. It suffices to show that (8.1)–(8.3) imply (7.2). Suppose that there exists an $N \in \langle D \rangle$ such that $F(\operatorname{co} N) \not\subset G(N)$; that is, there exist an $x \in \operatorname{co} N$ and $y \in Fx$ such that $y \notin Gz$ for all $z \in N$. In other words, by (8.1), $y \notin Hz$ for all $z \in N$, and hence $z \in X \backslash H^- y$. By (8.3), $\operatorname{co} N \subset X \backslash H^- y$. Since $x \in \operatorname{co} N$, we have $x \notin H^- y$ or $y \notin Hx$. Since $y \in Fx$, this contradicts (8.2). This completes our proof.

Here, in order to show that Theorems 5-8 are equivalent, the following suffices:

Proof of Theorem 5 using Theorem 8. Suppose that $Fx \cap Tx = \emptyset$ for all $x \in X$. Let $Gx = Y \backslash Sx$ for $x \in D$ and $Hx = Y \backslash Tx$ for $x \in X$. Then all of the requirements of Theorem 8 are satisfied. Therefore, there exists a $y_0 \in \overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\}$; that is, $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \notin Sx$ for all $x \in D$. This contradicts (5.3). This completes our proof.

REMARK. The first particular form of Theorem 8 is due to Tarafdar [**Tr2**] for $X = D = Y$ and $F = 1_X$. Horvath [**Ho**] also obtained a form of Theorem 8 for a space more general than convex spaces. Other particular forms are due to Guillerme [**Gu**], who applied them to obtain some minimax inequalities.

5. Minimax inequalities and geometric properties

There are many equivalent and useful formulations of Theorems 5–8 in the KKM theory. In this section, we give analytic alternatives, minimax inequalities, and geometric properties of convex sets.

We begin, in this section, with the following useful reformulation of Theorem 5:

THEOREM 9. *Let (X, D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, $A, B \subset Z$ sets, $f, g : X \times Y \to Z$ functions, and K a nonempty compact subset of Y. Suppose that*

(9.1) *for each $x \in D$, $\{y \in Y : g(x, y) \in A\}$ is compactly open and contained in $\{y \in Y : f(x, y) \in B\}$;*
(9.2) *for each $y \in F(X)$, $\{x \in X : f(x, y) \in B\}$ is D-convex; and*
(9.3) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that for each $y \in F(L_N)\backslash K$, there exists an $x \in L_N \cap D$ satisfying $g(x, y) \in A$.*

Then either

(i) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $g(x, \hat{y}) \notin A$ for all $x \in D$; or*
(ii) *there exists an $(\hat{x}, \hat{y}) \in F$ such that $f(\hat{x}, \hat{y}) \in B$.*

Proof of Theorem 9 using Theorem 5. Consider the multifunctions $S : D \to 2^Y$ and $T : X \to 2^Y$ given by

$$Sx = \{y \in Y : g(x, y) \in A\} \quad \text{for} \quad x \in D$$

and

$$Tx = \{y \in Y : f(x, y) \in B\} \quad \text{for} \quad x \in X.$$

Then (9.1), (9.2), and (9.3) imply (5.1), (5.2), and (5.4), resp. Suppose (i) does not hold. Then, for each $y \in \overline{F(X)} \cap K$ there exists an $x \in D$ such that $g(x, y) \in A$; that is, $\overline{F(X)} \cap K \subset S(D)$. Hence (5.3) holds. Therefore, by Theorem 5, F and T have a coincidence point; that is, (ii) holds.

REMARK. In our previous work [**P8**], a particular form of Theorem 9 has been given to derive Theorem 3(iii) for $\mathbb{V}$. In fact, Theorem 3(iii) follows from Theorem 9 with $X = D$ as follows:

Proof of Theorem 3(iii) using Theorem 9. Suppose that F is u.s.c. Let Y be a compact set such that $F(X) \subset Y \subset X$, and V an open convex neighborhood of the origin of E. We apply Theorem 9 with $X = D$, $Y = K$, $A = B = V$, $Z = E$, and $f = g : X \times Y \to E$ defined by $f(x, y) = y - x$ for $(x, y) \in X \times Y$. Then all of the requirements are satisfied. However, Condition (i) does not hold. Now, Condition (ii) implies the assertion $(*)$ in the proof of Theorem 3. This suffices for the existence of fixed point of F as in the proof of Theorem 3(iii).

PARTICULAR FORMS. 1. The first form of Theorem 9 is due to Lassonde [**L1**, Theorem 1.1′]. Note that if F is single-valued, then Y is not necessarily Hausdorff. Lassonde used his result to generalize earlier works of Iohvidov [**I**], Fan [**F4**], and Browder [**Bw2**]. Applications of this kind of results to the Tychonoff fixed point theorem and the study of invariant subspaces of certain linear operators are given in [**I**], [**F4**].

2. For $X = D$ and $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, Theorem 9 reduces to [**P6**, Theorem 5].

From Theorem 9, we have the following analytic alternative, which is a basis of various minimax inequalities.

THEOREM 10. *Let (X, D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, $\alpha \geq \beta$, $f : X \times Y \to \overline{\mathbb{R}}$, $g : D \times Y \to \overline{\mathbb{R}}$ extended real-valued functions, and K a nonempty compact subset of Y. Suppose that*

(10.1) *$g(x, y) \leq f(x, y)$ for all $(x, y) \in D \times Y$;*
(10.2) *for each $x \in D$, $\{y \in Y : g(x, y) > \alpha\}$ is compactly open;*
(10.3) *for each $y \in F(X)$, $\{x \in X : f(x, y) > \beta\}$ is D-convex ; and*
(10.4) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that, for each $y \in F(L_N) \backslash K$, there exists an $x \in L_N \cap D$ satisfying $g(x, y) > \alpha$.*

Then either

(i) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $g(x, \hat{y}) \leq \alpha$ for all $x \in D$; or*
(ii) *there exists an $(\hat{x}, \hat{y}) \in F$ such that $f(\hat{x}, \hat{y}) > \beta$.*

Proof. Put $Z = \overline{\mathbb{R}}$, $A = (\alpha, \infty]$, and $B = (\beta, \infty]$ in Theorem 9.

PARTICULAR FORMS. 1. The first form of Theorem 10 is due to Ben-El-Mechaiekh *et al.* [**BDG1,2**] for $X = D = Y = K$ and $F = 1_X$. The authors used their result to variational inequalities of Hartman-Stampacchia and Browder, a generalization of the Ky Fan minimax inequality, and others.

2. See also Fan [**F6**], Brézis, Nirenberg, and Stampacchia [**BNS**], Allen [**Al**], Granas [**Gr1,2**], Tan [**T**], Lin [**Ln**], Ko and Tan [**KT**], Deguire and Granas [**DG**], Takahashi [**Tk2**], Shih and Tan [**ST6**], Ding and Tan [**DT**], Ben-El-Mechaiekh [**Bn2**], Deguire [**D**], and Park [**P6**].

From Theorem 10, we clearly have the following generalization of the Ky Fan minimax inequality:

THEOREM 11. *Under the hypothesis of Theorem 10, if* $\alpha = \beta = \sup\{f(x,y) : (x,y) \in F\}$, *then*

(a) *there exists a* $\hat{y} \in \overline{F(X)} \cap K$ *such that*

$$g(x,\hat{y}) \leq \sup_{(x,y)\in F} f(x,y) \quad \textit{for all} \quad x \in D;$$

and (b) *we have the minimax inequality:*

$$\min_{y\in K} \sup_{x\in D} g(x,y) \leq \sup_{(x,y)\in F} f(x,y).$$

In order to show that Theorem 11 is equivalent to any of Theorems 1–10, we give the following:

Proof of Theorem 8 using Theorem 11. Define functions $g : D\times Y \to \mathbb{R}$ and $f : X \times Y \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} 0 & \text{if} \quad y \in Gx \\ 1 & \text{otherwise} \end{cases}$$

for $(x,y) \in D \times Y$ and

$$f(x,y) = \begin{cases} 0 & \text{if} \quad y \in Hx \\ 1 & \text{otherwise} \end{cases}$$

for $(x,y) \in X \times Y$. Put $\alpha = \beta = 0$. Then (7.1) and (7.3) imply (10.2) and (10.4), resp. Moreover, (8.1) implies (10.1) clearly. Further, (8.2) and (8.3) imply (10.3). In fact, for each $y \in F(X)$, we have $y \in H(X)$ by (8.2) and hence $y \in Hx$ for some $x \in X$. Then $X\backslash H^{-}y = \{x \in X : f(x,y) = 1\} = \{x \in X : f(x,y) > 0\}$ is D-convex by (8.3). Therefore, by Theorem 11, there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that

$$g(x,\hat{y}) \leq \sup_{(x,y)\in F} f(x,y) \quad \text{for all} \quad x \in D.$$

However, $\sup\{f(x,y) : (x,y) \in F\} \leq \sup\{f(x,y) : (x,y) \in H\} = 0$ by (8.2) and the definition of f. Hence $g(x,\hat{y}) = 0$ for all $x \in D$; that is, $\hat{y} \in Gx$ for all $x \in D$. Therefore,

$$\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset.$$

This completes our proof.

REMARK. Conclusion (b) can be written as follows:

$$\min_{y\in K}\sup_{x\in D} g(x,y) \le \inf_{F\in\mathfrak{A}_c^\kappa(X,Y)} \sup_{(x,y)\in F} f(x,y).$$

PARTICULAR FORMS. 1. Theorem 11 originates from the Ky Fan minimax inequality [**F6**] for $X = D = Y = K$, $f = g$, and $F = 1_X$. Fan applied his inequality to fixed point theorems, sets with convex sections, and potential theory. Later, the inequality became an important tool in nonlinear functional analysis, game theory, and economic theory.

2. In [**P6**, Theorem 9], we obtained Theorem 11 for $X = D$ and $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$. This includes earlier works of Fan [**F6,9**], Brézis, Nirenberg, and Stampacchia [**BNS**], Takahashi [**Tk1,2**], Yen [**Y**], Aubin [**Au**], Ben-El-Mechaiekh *et al.* [**BDG1-3**], Tan [**T**], Shih and Tan [**ST1,2**], Aubin and Ekeland [**AE**], Lassonde [**L1**], Granas and Liu [**GL1,2**], Lin [**Ln**], Ha [**Ha1,2**], and Park [**P2**].

The KKM theorem and the whole intersection theorem can be reformulated to minimax inequalities.

The following minimax inequality is equivalent to Theorem 7.

THEOREM 12. *Let (X,D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X,Y)$, and K a nonempty compact subset of Y. Let $\phi : D\times Y \to \overline{\mathbb{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbb{R}}$ such that*

(12.1) *for each $x\in D$, $\{y\in Y : \phi(x,y)\le\gamma\}$ is compactly closed;*

(12.2) *for each $N\in\langle D\rangle$ and $y\in F(\mathrm{co}\,N)$, $\min\{\phi(x,y) : x\in N\}\le\gamma$; and*

(12.3) *for each $N\in\langle D\rangle$, there exists a compact D-convex subset L_N of X containing N such that, for each $y\in F(L_N)\backslash K$, there exists an $x\in L_N\cap D$ satisfying $\phi(x,y)>\gamma$.*

Then (a) *there exists a $\hat{y}\in\overline{F(X)}\cap K$ such that*

$$\phi(x,\hat{y})\le\gamma \quad \text{for all} \quad x\in D;$$

and (b) *if $\gamma = \sup\{\phi(x,y) : (x,y)\in F\}$, then we have the minimax inequality:*

$$\min_{y\in K}\sup_{x\in D}\phi(x,y)\le \sup_{(x,y)\in F}\phi(x,y).$$

Proof of Theorem 12 using Theorem 7. Let $Gx = \{y \in Y : \phi(x,y) \leq \gamma\}$ for $x \in D$. Then (12.1) and (12.3) imply (7.1) and (7.3) clearly. We show that (12.2) implies (7.2). Suppose that there exists an $N \in \langle D \rangle$ such that $F(\operatorname{co} N) \not\subset G(N)$. Choose a $y \in F(\operatorname{co} N)$ such that $y \notin G(N)$, whence $\phi(x,y) > \gamma$ for all $x \in N$. Then $\min_{x\in N} \phi(x,y) > \gamma$, which contradicts (12.2). Therefore, by Theorem 7, there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $\hat{y} \in Gx$ for all $x \in D$; that is, $\phi(x,\hat{y}) \leq \gamma$ for all $x \in D$. This completes the proof of (a). Note that (b) clearly follows from (a).

Proof of Theorem 7 using Theorem 12. Define $\phi : D \times Y \to \overline{\mathbb{R}}$ by

$$\phi(x,y) = \begin{cases} 0 & \text{if} \quad y \in Gx \\ 1 & \text{otherwise} \end{cases}$$

for $(x,y) \in D \times Y$. Put $\gamma = 0$ in Theorem 12. Then (7.1) clearly implies (12.1). We show that (7.2) implies (12.2). In fact, suppose that there exist an $N \in \langle D \rangle$ and $y \in F(\operatorname{co} N)$ such that $\min\{\phi(x,y) : x \in N\} > 0$. Then $y \notin Gx$ for all $x \in N$; that is, $F(\operatorname{co} N) \not\subset G(N)$, which contradicts (7.2). Moreover, we show that (7.3) implies (12.3). In fact, for $y \in F(L_N)\backslash K$, we have $y \notin Gz$ for some $z \in L_N \cap D$; that is, $\phi(z,y) > 0$. Therefore, all of the requirements of Theorem 12 are satisfied. Hence, there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $\phi(x,\hat{y}) = 0$ for all $x \in D$; that is, $\hat{y} \in \bigcap\{Gx : x \in D\}$. This completes our proof.

REMARK. In the proof of the equivalency of Theorems 7 and 12 we used the fact that (12.2) is equivalent to

(12.2)′ the map $x \mapsto Gx = \{y \in Y : \phi(x,y) \leq \gamma\}$ satisfies Condition (7.2).

For similar arguments, see Ding, Kim, and Tan [**DKT**] and Chang and Zhang [**CZ**].

PARTICULAR FORMS. The first particular forms of Theorem 12 are due to Zhou and Chen [**ZC**, Theorem 2.11 and Corollary 2.13] for $X = D = Y = K$ and $F = 1_X$. Those results are applied to obtain a variation of the Ky Fan inequality, a saddle point theorem, and a quasi-variational inequality.

In 1961, Fan [**F2**] gave a "geometric" lemma which is the geometric equivalence of his version of the KKM theorem. In many of his works in the KKM theory, Fan actually based his arguments mainly on the geometric property of a convex space. We now deduce two geometric forms of Theorem 7. The first one is as follows:

THEOREM 13. *Let (X, D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $A \subset B \subset C \subset X \times Y$. Suppose that*

(13.1) *for each $x \in D$, $\{y \in Y : (x, y) \in C\}$ is compactly closed in Y;*
(13.2) *for each $y \in F(X)$, $\{x \in X : (x, y) \notin B\}$ is D-convex;*
(13.3) *A is the graph of F; and*
(13.4) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap_{x \in L_N \cap D}\{y \in Y : (x, y) \in C\} \subset K$.*

Then there exists a $y_0 \in \overline{F(X)} \cap K$ such that $D \times \{y_0\} \subset C$.

Proof of Theorem 13 using Theorem 7. For each $x \in D$, let

$$Gx = \{y \in Y : (x, y) \in C\},$$

which is compactly closed by (13.1). Moreover, for each $N \in \langle D \rangle$, we have

$$F(\operatorname{co} N) \subset G(N).$$

In fact, let $y \in F(\sum_{i=1}^n \lambda_i x_i)$ with $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^n \lambda_i = 1$, and $N = \{x_1, x_2, \ldots, x_n\} \in \langle D \rangle$. If $y \notin G(N)$, then $(x_i, y) \notin C$ for each i. Since $B \subset C$, we have $(x_i, y) \notin B$ for each i. By (13.2), $\{x \in X : (x, y) \notin B\}$ is D-convex, and hence $(\sum_{i=1}^n \lambda_i x_i, y) \notin B$. Since $A \subset B$, we have $(\sum_{i=1}^n \lambda_i x_i, y) \notin A$; that is, $y \notin F(\sum_{i=1}^n \lambda_i x_i)$ by (13.3), which is a contradiction. Since (13.4) clearly implies (7.3), G satisfies all of the requirements of Theorem 7. Therefore, we have

$$\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset.$$

Hence, there exists a $y_0 \in \overline{F(X)} \cap K$ such that $y_0 \in \bigcap\{Gx : x \in D\}$; that is, $D \times \{y_0\} \subset C$.

PARTICULAR FORMS. 1. The original form of Theorem 13 due to Fan [**F2**] is the case $X = D = Y$, $A = B = C$, and $F = 1_X$. Fan's geometric lemma has many applications, among which are fixed point theorems, theorems on minimax, existence of equilibrium points, extensions of monotone sets, a fundamental existence theorem in potential theory, variational inequalities, and many others.

2. For $X = D$ and $F \in \mathbb{V}(X, Y)$, Theorem 13 reduces to Park [**P9**, Theorem 12], which extends earlier results of Fan [**F2,5,6**], Takahashi [**Tk1**], Shih and Tan [**ST2**], Lin [**Ln**], Park [**P2,6**], Ha [**H1**], Shioji [**So**], and Sehgal, Singh, and Whitfield [**SSW**].

The following form of Theorem 13 is also widely used in the KKM theory:

THEOREM 14. *Let (X, D) be a convex space, Y a Hausdorff space, K a nonempty compact subset of Y, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $C \subset B \subset A \subset X \times Y$. Suppose that*

(14.1) *for each $x \in D$, $\{y \in Y : (x, y) \in C\}$ is compactly open in Y;*
(14.2) *for each $y \in F(X)$, $\{x \in X : (x, y) \in B\}$ is D-convex;*
(14.3) *for each $y \in \overline{F(X)} \cap K$, there exists an $x \in D$ such that $(x, y) \in C$; and*
(14.4) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap_{x \in L_N \cap D} \{y \in Y : (x, y) \notin C\} \subset K$.*

Then there exist an $x_0 \in X$ and a $y_0 \in Fx_0$ such that $(x_0, y_0) \in A$.

Proof of Theorem 14 using Theorem 13. Consider Theorem 13 with (A^c, B^c, C^c) instead of (A, B, C). Then (13.1), (13.2), and (13.4) are satisfied automatically. Since (14.3) is the negation of the conclusion of Theorem 13, we should have the negation of (13.3). Therefore, the conclusion follows.

Proof of Theorem 5 using Theorem 14. Let $A = B$ be the graph of T and C the graph of S. Then (5.1)–(5.4) imply (14.1)–(14.4). Therefore, by Theorem 14, there exists an $(x_0, y_0) \in F$ such that $(x_0, y_0) \in A$; that is, F and T have a coincidence point.

Consequently, Theorems 1–14 are all equivalent to each other.

PARTICULAR FORMS. 1. The origin of Theorem 14 goes back to Fan [**F6**] for $X = D = Y = K$, $A = B = C$, and $F = 1_X$. This is equivalent to the Fan-Browder fixed point theorem [**Bw1-3**].

2. For $X = D$ and $F \in \mathbb{V}(X, Y)$, Theorem 14 reduces to [**P9**, Theorem 13], which extends earlier works of Fan [**F7**], Shih and Tan [**ST2,4,6**], and Park [**P2,4**].

Finally, there have appeared many types of "generalized" convex spaces and the KKM type theorems for those spaces. Some of the results in this paper can be modified for those new spaces.

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