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DOUBLE ZEROS OF THE SZEGŐ KERNEL

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1. Introduction

When we study complex analysis, we often encounter multi-valued functions. Provably the best known example would be $\sqrt{z}$, which has a singularity at the origin. In general, let Ω_1 and Ω_2 be domains in $\mathbf{C}$, and let Γ be a subvariety of $\Omega_1 \times \Omega_2$. Let $\pi_i : \Gamma \to \Omega_i$ be the projections, $i = 1, 2$. Then the multi-valued map

$$f = \pi_2 \pi_1^{-1} : \Omega_1 \to \Omega_2$$

is called a *holomorphic correspondence.* If π_1 and π_2 are both proper, then f is called *proper.* K. Stein pioneered in the study of proper holomorphic correspondences (See [7]). In recent years, E. Bedford and S. Bell studied in this area (See [2], [3]).

In this paper, we want to study the zero set of a Szegő kernel, which gives a proper self-correspondence of a domain. It is a natural object of the domain in a sense that it is preserved by a biholomorphic map. All domains in this paper are assumed to be bounded, finitely connected in $\mathbf{C}$ with nondegenerate boundary. Let Ω be a domain with C^1 boundary. Let $A(\Omega) = C(\overline{\Omega}) \cap H(\Omega)$. Let $H^2(b\Omega)$ be the closure of $A(b\Omega) = A(\Omega)|_{b\Omega}$ in the $L^2(b\Omega)$ space. Then the orthogonal projection $P : L^2(b\Omega) \to H^2(b\Omega)$ is called the *Szegő projection.* Each element f of $H^2(b\Omega)$ has a holomorphic extension to Ω. The *Szegő kernel* $S(z,w)$ is defined to be the kernel function that satisfies

$$(Pf)(z) = \int_{b\Omega} S(z,\zeta) f(\zeta) d\sigma(\zeta),$$

for $z \in \Omega$, $f \in L^2(b\Omega)$. It is well-known that $S(z,w) = \overline{S(w,z)}$ and $S(z,w)$ is holomorphic in the first variable. The zero set of the Szegő

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kernel is called the *Szegő variety*. The Szegő variety is defined to be *irreducible* if the analytic subvariety $\{(z,\overline{w}); (z,w) \in$ Szegő variety$\}$ is irreducible. Suppose $f : \Omega_1 \to \Omega_2$ is biholomorphic and Ω_1, Ω_2 have C^1 boundary and f, f' extend continuously to $\overline{\Omega_1}$. The Szegő kernels of two domains are related by the formula

$$S_1(z,w) = \sqrt{f'(z)} S_2(f(z), f(w)) \overline{\sqrt{f'(w)}} \tag{1}$$

for $z, w \in \overline{\Omega_1} \times \overline{\Omega_1} - \mathcal{D}$, where $\mathcal{D} = \{(z,z); z \in b\Omega_1\}$. Since f' is never equal to zero for $z \in \Omega_1$, the above equation shows that the Szegő variety is invariant under a biholomorphic map.

Suppose Ω is a n-connected domain with real analytic boundary. For any fixed $w \in \Omega$, $S_w(z) = S(z,w)$ has $n-1$ zeros in Ω, counting with multiplicity. If O is a sufficiently small neighborhood of $\overline{\Omega}$, then $S_w(z)$ is a well-defined holomorphic function on O and it does not have a zero on $O - \overline{\Omega}$. Denote the zeros of $S_w(z)$ by $Z_1(w), \ldots, Z_{n-1}(w)$. Then the multi-valued map

$$w \to Z_1(w), \ldots, Z_{n-1}(w)$$

is a proper *anti-holomorphic correspondence*, which means that it is a complex conjugate of a proper holomorphic correspondece. Let us denote the boundary curves of Ω by $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$. When w converges to γ_0, all the zeros of $S(z,w)$ become simple zeros. Moreover, for $j = 0, 1, \ldots, n-1$, we can find neighborhoods O_j's of γ_j's so that $Z_k : \Omega_0 \to \Omega_k$ is a *conjugate-biholomorphic map* (in other words, it is a complex conjugate of a biholomorphic map), and $Z_k(\gamma_0) = \gamma_k$ for each $k = 1, \ldots, n-1$. For the proof of the above facts, see [5]. When Ω is 2-connected, Z_1 is a conjugate-biholomorphic map from Ω to itself. So, our main interest is the case $n \geq 3$.

2. Double zeros of the Szegő kernel

In this chapter, our goal is to prove Theorem 1, our main theorem. Later, we wil see that we can weaken the hypothesis of Theorem 1 considerably.

Suppose Ω is a n-connected domain ($n \geq 2$) with real analytic boundary curves

$$\gamma_0, \gamma_1, \ldots, \gamma_{n-1}.$$

Let $\gamma = \gamma_0 + \cdots + \gamma_{n-1}$. Let $S'(z,w) = \partial S(z,w)/\partial z$. Define

$$g_k(w) = \frac{1}{2\pi i} \int_\gamma z^k \frac{S'(z,w)}{S(z,w)} dz \tag{2}$$

for $w \in \Omega$, for $k = 1, \ldots, n-1$. Since $S(z,w)$ and $S'(z,w)$ are anti-holomorphic in w, $g_k(w)$'s are anti-holomorphic functions. By the argument principle, we have $g_k(w) = Z_1(w)^k + \cdots + Z_{n-1}(w)^k$. Since each $Z_j(w)$ extends anti-holomorphically to some neighborhood of γ_i, $g_k(w)$ also extends. By Newton's identity, any symmetric polynomial in $Z_j(w)$'s can be expressed as a polynomial in $g_k(w)$'s.

Defind the *Szegő polynomial* $H(z,w)$ by

$$H(z,w) = \prod_{j=1}^{n-1} (z - Z_j(w)) = z^{n-1} + h_{n-2}(w) z^{n-2} + \cdots + h_0(w).$$

Each $h_l(w)$ is a polynomial in $g_k(w)$'s, so $h_l(w)$ s are anti-holomorphic in some neighborhood of $\overline{\Omega}$. The Szegő kernel $S(z,w)$ and the Szegő polynomial $H(z,w)$ define the same zero set in $\Omega \times \Omega$. The *discriminant* $\delta(w)$ of $H(z,w)$ is defined by $\delta(w) = \prod_{i<j}(Z_i(w) - Z_j(w))^2$. It is well-known that the discriminant of a polynomial can be expressed by addition and multiplication of its coefficients. So $\delta(w)$ is an anti-holomorphic function in some neighborhood of $\overline{\Omega}$. If $\delta(w_0) = 0$ for some $\delta(w_0) \in \Omega$, then $S(z, w_0)$ would have a double zero. When $w \in b\Omega$, $Z_j(w)$'s are all different from each other. So $\delta(w) \neq 0$ for $w \in b\Omega$. By calculating the argument increase or decrease along the boundary, we can find out the number of zeros of δ in Ω.

THEOREM 1. *Suppose Ω is an n-connected domain with real analytic boundary in* $\mathbf{C}$, *$n \geq 3$. Then the discriminant $\delta(w)$ has $2(n-1)(n-2)$ zeros in Ω, counting with multiplicity.*

We denote the boundary curves of Ω by $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$. We assume that γ_0 is the outer boundary. We give the usual orientation to the boundary curves, so we trace γ_0 counter-clockwise, while $\gamma_1, \ldots, \gamma_{n-1}$, clockwise. We need two lemmas to prove theorem 1. For convenience, we will calculate the number of zeros of $\overline{\delta}(w)$, the complex conjugate of $\delta(w)$.

LEMMA 2. *With the above assumptions,* $\int_{\gamma_0} \overline{\delta}'(w)/\overline{\delta}(w)dw = 0$.

Proof. By the Leibniz rule,

$$\int_{\gamma_0} \frac{\overline{\delta}'(w)}{\overline{\delta}(w)} dw = 2 \sum_{1 \le i < j \le n-1} \int_{\gamma_0} \frac{(\overline{Z}_i(w) - \overline{Z}_j(w))'}{(\overline{Z}_i(w) - \overline{Z}_j(w))} dw.$$

When w moves along γ_0 counter-clockwise, $\overline{Z}_i(w)$ moves along $\overline{\gamma}_i$ clockwise for $i = 1, \ldots, n-1$. For any fixed point z_i on $\overline{\gamma}_i$, we can define a single-valued log function on $z_i - \overline{\gamma}_j$. This is posible because the curve $\overline{\gamma}_j$ does not circle around z_i. Similarly, for any point z_j on $\overline{\gamma}_j$, we can define a single-valued log function on $\overline{\gamma}_i - z_j$. Fix a point w_0 on γ_0. Now, define $\mathrm{Log}(\overline{Z}_i(w) - \overline{Z}_j(w))$ on γ_0 by

$$\begin{aligned} &\mathrm{Log}(\overline{Z}_i(w) - \overline{Z}_j(w)) \\ &= \log(\overline{Z}_i(w_0) - \overline{Z}_j(w_0)) \\ &\quad + (\log(\overline{Z}_i(w) - \overline{Z}_j(w_0)) - \log(\overline{Z}_i(w_0) - \overline{Z}_j(w_0))) \\ &\quad + (\log(\overline{Z}_i(w) - \overline{Z}_j(w)) - \log(\overline{Z}_i(w) - \overline{Z}_j(w_0))) \end{aligned}$$

Here,

$$\log(\overline{Z}_i(w) - \overline{Z}_j(w_0)) - \log(\overline{Z}_i(w_0) - \overline{Z}_j(w_0))$$

is defined as a log function on $\overline{\gamma}_i - \overline{Z}_j(w_0)$, where $\overline{Z}_j(w_0)$ is a fixed point on γ_j. Similarly,

$$\log(\overline{Z}_i(w) - \overline{Z}_j(w)) - \log(\overline{Z}_i(w) - \overline{Z}_j(w_0))$$

is defined as a log function on $\overline{Z}_i(w) - \overline{\gamma}_j$, where $\overline{Z}_i(w)$ is a fixed point on $\overline{\gamma}_i$. Since the log functions on $\overline{\gamma}_i - z_j$ and $z_i - \overline{\gamma}_j$ are single-valued, when w circles around γ_0 and returns to the starting point, $\mathrm{Log}(\overline{Z}_i(w) - \overline{Z}_j(w))$ also returns to the same value. So $\mathrm{Log}(\overline{Z}_i - \overline{Z}_j)$ is a single-valued function on γ_0. Hence,

$$\int_{\gamma_0} \frac{(\overline{Z}_i(w) - \overline{Z}_j(w))'}{(\overline{Z}_i(w) - \overline{Z}_j(w))} dw = [\mathrm{Log}(\overline{Z}_i - \overline{Z}_j)]_{w_0}^{w_0} = 0.$$

Thus, $\int_{\gamma_0} \overline{\delta}'(w)/\overline{\delta}(w)dw = 0$.

LEMMA 3. *With the same assumptions, for $l = 1, \dots, n-1$,*

$$\int_{\gamma_l} \frac{\overline{\delta}'(w)}{\overline{\delta}(w)} dw = 2(n-2)(2\pi i).$$

Proof. For $w \in \gamma_l$, we denote zeros of $S_w(z)$ by

$$Z_0(w), \dots, Z_{l-1}(w), Z_{l+1}(w), \dots, Z_{n-1}(w),$$

where each $Z_j(w)$ is on γ_j. By the Leibniz rule,

$$\int_{\gamma_l} \frac{\overline{\delta}'(w)}{\overline{\delta}(w)} dw = 2 \sum_{\substack{0 \le j < k \le n-1 \\ j,k \ne l}} \int_{\gamma_l} \frac{(\overline{Z}_j(w) - \overline{Z}_k(w))'}{(\overline{Z}_j(w) - \overline{Z}_k(w))} dw.$$

As in the proof of lemma 2, for all $j, k > 0$, we can define a single-valued log function $\operatorname{Log}(\overline{Z}_j - \overline{Z}_k)$ on γ_l. So,

$$\int_{\gamma_l} \frac{(\overline{Z}_j(w) - \overline{Z}_k(w))'}{(\overline{Z}_j(w) - \overline{Z}_k(w))} dw = 0$$

for all $j, k > 0$.

Now, for any fixed point z_k on $\overline{\gamma}_k$, we define a log function on $\overline{\gamma}_0 - z_k$. Since $\overline{\gamma}_0$ circles around z_k, this log function is not single-valued. It increases by $2\pi i$ when a point circles around $\overline{\gamma}_0$ once. Fix a point w_0 on γ_l. Define $\operatorname{Log}(\overline{Z}_0(w) - \overline{Z}_k(w))$ on γ_l by

$$\begin{aligned}
&\operatorname{Log}(\overline{Z}_0(w) - \overline{Z}_k(w)) \\
&= \log(\overline{Z}_0(w_0) - \overline{Z}_k(w_0)) \\
&\quad + (\log(\overline{Z}_0(w) - \overline{Z}_k(w_0)) - \log(\overline{Z}_0(w_0) - \overline{Z}_k(w_0))) \\
&\quad + (\log(\overline{Z}_0(w) - \overline{Z}_k(w)) - \log(\overline{Z}_0(w) - \overline{Z}_k(w_0))).
\end{aligned}$$

Remember that

$$\log(\overline{Z}_0(w) - \overline{Z}_k(w_0)) - \log(\overline{Z}_0(w_0) - \overline{Z}_k(w_0))$$

is defined as a log function on $\overline{\gamma}_0 - \overline{Z}_k(w_0)$, and it is not single-valued. It increases by $2\pi i$. While

$$\log(\overline{Z}_0(w) - \overline{Z}_k(w)) - \log(\overline{Z}_0(w) - \overline{Z}_k(w_0))$$

is defined as a log function on $\overline{Z}_0(w) - \overline{\gamma}_k$, and it is single-valued. So, $\operatorname{Log}(\overline{Z}_0(w) - \overline{Z}_k(w))$ will increase by $2\pi i$ when w circles around γ_l once. Hence,

$$\int_{\gamma_l} \frac{(\overline{Z}_0(w) - \overline{Z}_k(w))'}{(\overline{Z}_0(w) - \overline{Z}_k(w))} dw = \left[\operatorname{Log}(\overline{Z}_0(w) - \overline{Z}_k(w))\right]_{w_0}^{w_0} = 2\pi i.$$

Thus, $\int_{\gamma_l} \overline{\delta}'(w)/\overline{\delta}(w) dw = 2(n-2)(2\pi i)$.

Proof of theorem 1. By the above lemmas,

$$\int_{\gamma} \overline{\delta}'(w)/\overline{\delta}(w) dw = 2(n-1)(n-2)(2\pi i).$$

Since $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_{n-1}$ is homologous to zero in Ω, we can apply the argument principle. See p.152 of [1]. So $\overline{\delta}$ has $2(n-1)(n-2)$ zeros in Ω, counting with mutiplicity. So δ has $2(n-1)(n-2)$ zeros. The proof is completed.

It is well-known that for any finitely connected domain Ω_1 with non-degenerate boundary components, there exists a biholomorphic map $f : \Omega_1 \to \Omega_2$ so that Ω_2 is a bounded domain with real analytic boundary (See p.252 of [1]). When $b\Omega_1$ is C^k smooth, f and its derivatives of order up to $k-1$ extend continuously to $\overline{\Omega}_1$. Because of the formula (1) of chapter 1, Theorem 1 is true when $b\Omega$ is C^1. So, the Szegő kernel $S(z, w)$ of a domain Ω with connectivity $n \geq 3$ has a double zero at some point. When the boundary of Ω is so bad that the Szegő kernel is not defined at all, we can still get some results. find a domain Ω_2 with real analytic boundary which is biholomorphic to Ω, and apply Theorem 1 to Ω_2. It shows that any domain Ω with connectivity $n \geq 3$ has some special points, which are natural objects of the domain in a sense that it should be preserved by biholomorphic maps. We conclude this chapter with a lemma that we will need in the next chapter.

LEMMA 4. *Let Ω be a domain with connectivty $n \geq 2$. Let f be an automorphism of Ω that sends each boundary component to a different boundary component. Then $f - id$ has two zeros in Ω.*

Proof. We may assume that Ω has a real analytic boundary. Let $\gamma = \gamma_0 + \cdots + \gamma_{n-1}$ be the boundary components of Ω, where γ_0 is the outer boundary. We can calculate the argument increase of $f - id$ along γ. Use exactly the same method as that of Lemma 2 and lemma 3. Only on γ_0 and $f^{-1}(\gamma_0)$, $f - id$ has argument increase of $2\pi i$. So $f - id$ has two zeros in Ω, counting with multiplicity. The proof is completed.

3. When Ω is 3-connected.

All domains in this chapter are assumed to be 3-connected. Let Ω be a domain with the Szegő kernel $S(z, w)$. Let $\{Z_1, Z_2\}$ be the proper correspondence defined by $S(z, w)$. By the remarks following the proof of Theorem 1, we can find a point a in Ω so that Z_1 and Z_2 have the same value at a. We should consider two different cases.

(1) Z_1 and Z_2 are single-valued functions near a.

(2) Z_1 and Z_2 are not single-valued near a. By anti-analytic continuation around a, Z_1 becomes Z_2 and Z_2 becomes Z_1.

First, let us study the case (1). Let $b = Z_1(a) = Z_2(a)$. It is easy to see that $(\overline{\partial} Z_i/\partial\overline{z})(a) \neq 0$ for $i = 1, 2$. Indeed, if $(\overline{\partial} Z_i/\partial\overline{z})(a) = 0$, then the proper correspondence $\{Z_1, Z_2\} = \{Z_1^{-1}, Z_2^{-1}\}$ would map a point near b to more than two points near a, which is a contradiction. So, Z_1 and Z_2 are conjugate-biholomorphic near a. By the Theorem 1, $(Z_1 - Z_2)^2$ has 4 zeros. Since $Z_1 - Z_2$ has zeros at a and b, it cannot have any other zeros on Ω. So, Z_1 and Z_2 do not have any singular points, in other words, Z_1 and Z_2 are locally conjugate-bihlomorphic at any point.

Let $\gamma_0, \gamma_1, \gamma_2$ be the boundary components of Ω. Since Z_1 and Z_2 are conjugate-biholomorpic near γ_j for $j = 0, 1, 2$, we can define Z_1 and Z_2 as single-valued functions on Ω by the Monodromy Theorem. Since Z_1 and Z_2 are locally conjugate-biholomorphic, they should map Ω onto Ω. Since $\{Z_1, Z_2\}$ maps a point in Ω to two points, Z_1 and Z_2 should be one-to-one. So Z_1 and Z_2 are conjugate-biholomorphic self maps of Ω. We may assume that Z_1 maps γ_0 to γ_1, γ_1 to γ_2, γ_2 to γ_0. Then Z_2

should send γ_0 to γ_2, γ_2 to γ_1, γ_1 to γ_0. So, $Z_1 \circ Z_2 =$ identity because it is a biholomorphic map of Ω that does not move the boundary (See [6]). It is easy to see that $Z_1 \circ Z_1$ is an automorphism of order 3 with a and b as fixed points. Because of the Lemma 4, it does not have any other fixed point.

THEOREM 5. *Let Ω be a 3-connected domain. Suppose the Szegő variety of Ω is reducible. Then the zero functions Z_1, Z_2 are conjugate-biholomorphic self maps of Ω, and $Z_1 = Z_2^{-1}$. There exist two points a, b such that $Z_i(a) = b$, $Z_i(b) = a$ for $i = 1, 2$. $Z_i \circ Z_i$ is an automorphism of order 3 that has a, b as fixed points.*

Proof. This theorem follows by combining the above results.

Next, let us study the case (2). We need a lemma.

LEMMA 6. *Suppose Ω is a 3-connected domain. Let a be a point of Ω such that $S_a(z)$ has a double zero, say, b. If the Szegő variety is irreducible, then a is not a double zero of $S_b(z)$.*

Proof. Assume that a is a double zero of $S_b(z)$. Let

$$\sigma(t) : [0,1] \to \overline{\Omega}$$

be a curve that circles around a once in counter-clockwise direction and

$$\sigma(0) = \sigma(1) \in \gamma_0,\ \sigma(\frac{1}{3}) \in \gamma_1,\ \sigma(\frac{2}{3}) \in \gamma_2$$

where $\gamma_0, \gamma_1, \gamma_2$ are boundary components of Ω. Define $\sigma : [1,2] \to \overline{\Omega}$ by $\sigma(t+1) = \sigma(t)$. Then, the curve

$$Z_i \circ \sigma : [0,2] \to \overline{\Omega}$$

should circle around b twice clockwise. By considering the fact that $\{Z_1, Z_2\}$ should map a boundary component to different boundary components, we see that on some interval $[k/3, (k+1)/3]$, $Z_i \circ \sigma$ should move from some boundary component to the same boundary component. This is a contradiction to the fact that $\{Z_1, Z_2\}$ is the inverse of $\{Z_1, Z_2\}$ and each Z_i is conjugate-biholomorphic near the boundary components. The proof is completed.

When the Szegő variety is irreducible, we need to be a little bit careful to count the zeros of $Z_1 - Z_2$. The above lemma shows that when σ circles around a singular point a twice, $Z_i \circ \sigma$ circles around $Z_i(a)$ once. So Z_i behaves like $\sqrt{z}$ at the origin. Thus, at each singular points, $(Z_1 - Z_2)^2$ has one zero. So, there exist four singular points of $\{Z_1, Z_2\}$ in Ω.

LEMMA 7. *Suppose the Szegő variety of Ω is irreducible. Then Ω cannot have an automorphism of order 3.*

Proof. Suppose Ω has an automorphism f of order 3. Then f should send each boundary component to a different boundary component (See [6]). So f has at most 2 fixed points. Let a_1, a_2, a_3, a_4 be the singular points of the correspondence $\{Z_1, Z_2\}$. By the formula (1) of chapter 1, $\{a_i\}$ should be mapped to $\{a_i\}$ by f. Since f is of order 3, it should map at least one of a_j's, say a_1, to itself. Let b be the double zero of $S(z, a_1)$. Then, by the formula (1) of chapter 1, b should be a fixed point of f. Since a_1 is a simple zero of $S_b(z)$, $S_b(z)$ should have another zero, say c, in Ω. Then c should be a fixed point of f by the same reason. So f has 3 fixed points, which is a contradiction. So, there cannot exist an automorphism of order 3.

Combining these results, we have proved the following theorem.

THEOREM 8. *For a 3-connected domain Ω, the following statements are equivalent.*

(1) Ω has an automorphism of order 3.

(2) The Szegő variety of Ω is reducible.

(3) There exist two points a, b such that a is a double zero of $S_b(z)$, and b is a double zero of $S_a(z)$.

4. Epilog

3-connected domains that satisfy Theorem 8 have an interesting property. Let ϕ be a harmonic function on Ω continous up to the boundary and $\phi = 1$ on one of the boundary component and $\phi = 0$ on the other two boundary components. Then,

$$\phi(z) + \phi(Z_1(z)) + \phi(Z_2(z)) \equiv 1$$

on $\overline{\Omega}$. Put $z = a$, then we get $\phi(a) + \phi(b) + \phi(b) = 1$. Similarly, $\phi(b) + \phi(a) + \phi(a) = 1$. So, we have $\phi(a) = \phi(b) = 1/3$.

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