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ON THE GAUSS MAP OF QUADRIC HYPERSURFACES

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1. Introduction

Let M^n be a connected hypersurface in Euclidean $(n+1)$-space E^{n+1}, and let $G : M^n \to S^n(1) \subset E^{n+1}$ be its Gauss map. Then, according to a theorem of E.A. Ruh and J.Vilms [5], M^n is a surface of constant mean curvature if and only if as a map from M^n to $S^n(1)$, G is harmonic, or equivalently, if and only if

$$\Delta G = \|dG\|^2 G, \tag{1.1}$$

where Δ is the Laplace operator on M^n corresponding to the induced metric on M^n from E^{n+1} and where G is seen as a function from M^n to E^{n+1}. A special case of (1.1) is given by

$$\Delta G = \lambda G, \; (\lambda \in R) \tag{1.2}$$

that is, the case where the Gauss map $G : M^n \to E^{n+1}$ is an eigenfunction of the Laplacian Δ on M^n. And such hypersurfaces satisfying (1.2) were classified for some cases in [4].

On the other hand, F.Dillen, J. Pas and L. Verstraelen [3] proved that among the surfaces of revolution in E^3, the only ones whose Gauss map satisfy the condition

$$\Delta G = AG, \; (A \in R^{3\times 3}) \tag{1.3}$$

are the planes, the spheres and the circular cylinders. And C. Baikoussis and D.E. Blair [1] recently proved that among the ruled surfaces

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in E^3, the only ones whose Gauss map satisfy (1.3) are the planes and the circular cylinders.

There are hyperplanes, hyperspheres and the cylinders over round spheres which satisfy the condition

$$\Delta G = AG, \quad (A \in R^{(n+1)\times(n+1)}). \tag{1.4}$$

And those examples are quadric hypersurfaces in E^{n+1}.

A question which arises now is : Are there any other quadric hypersurfaces in E^{n+1} satisfying condition (1.4)?

In particular, we will prove the following:

THEOREM. *Among the quadric hypersurfaces in E^{n+1}, the only ones whose Gauss map satisfy (1.4) are the hyperplanes, the hyperspheres and the cylinders over round spheres.*

Our proof of the above theorem essentially follows a reasoning which is given in [2], where B.Y. Chen, F.Dillen and H.Z. Song classified the quadirc hypersurfaces of finite type.

2. Examples and preliminaries

(1) hyperplane. In this case G is constant, so $\Delta G = 0$ and the hyperplane satisfies (1.4) with $A = 0$.

(2) sphere. Let $S^n(r)$ be the sphere with center 0 and radius r. If x denotes the position vector field of $S^n(r)$, then the Gauss map G is given by $\frac{1}{r}x$. Since $\Delta x = -nH$ and $H = -\frac{1}{r}G$, where H is the mean curvature vector field on $S^n(r)$, we have $\Delta G = \frac{1}{r^2}G$. Hence we find that $S^n(r)$ satisfies (1.4) with $A = \text{diag}\,(1/r^2, \cdots, 1/r^2)$.

(3) cylinder over a round sphere. We consider the hypersurface $M = S^p(r) \times R^{n-p}$. Then as in the case of sphere, we have $\Delta G = AG$ with

$$A = \text{diag}(1/r^2, \cdots, 1/r^2,\ 0, \cdots, 0)\,(\text{with } n-p \text{ zeros }).$$

Let M^n be a hypersurface in the Euclidean space E^{n+1}. We denote by G, A, σ and α the Gauss map of M^n, the Weingarten map, the second fundamental form and the mean curvature of M^n with respect to G defined by $\alpha = \frac{1}{n} < \mathrm{tr}(\sigma), G >$. Then we have the following ([4]):

$$\Delta G = n\nabla\alpha + |A|^2 G, \tag{2.1}$$

where $|A|^2$ is defined by $\mathrm{tr}(A^2)$.

If $\Delta G = 0$, then $|A|^2 = \mu_1^2 + \cdots + \mu_n^2 = 0$, where $\mu_1, \cdots, \mu_n$ are principal curvature of M^n with respect to G. Hence M^n is totally geodesic and we obtain the following:

LEMMA 1. *The hyperplanes are the only hypersurfaces satisfying* $\Delta G = 0$.

3. Quadric hypersurfaces

A subset M of an $(n+1)$-dimensional Euclidean space E^{n+1} is called a quadric hypersurface if it is the set of points $(x_1, \cdots, x_{n+1})$ satisfying the following equation of the second degree:

$$\sum_{i,j=1}^{n+1} a_{ij}x_ix_j + \sum_{i=1}^{n+1} b_ix_i + c = 0, \tag{3.1}$$

where a_{ij}, b_i, c are all real numbers. Suppose that M is not a hyperplane. Then A is not a zero matrix and we may assume without loss of generality that the matrix $A = (a_{ij})$ is symmetric. By applying a coordinate transformation in E^{n+1} if necessary, we may assume that (3.1) takes one of the following canonical forms:

(I) $\sum_{i=1}^{r} a_ix_i^2 + 2x_{r+1} = 0,$

(II) $\sum_{i=1}^{r} a_ix_i^2 + 1 = 0,$

(III) $\sum_{i=1}^{r} a_ix_i^2 = 0,$

where $(a_1, \cdots, a_r, 0, \cdots, 0)$ is proportional to the eigenvalues of the matrix A with $a_1 a_2 \cdots a_r \neq 0$. In the cases where $r = n$ in (I) and $r = n + 1$ in (II) and (III) the hypersurface is called a properly n-dimensional quadric hypersurface, and in other cases, a quadric cylindrical hypersurface. In the case (I), the quadric cylindrical hypersurface is the product of an $(n - r)$-dimensional linear subspace and a properly r- dimensional quadric hypersurface. In case (II) and (III), the quadric cylindrical hypersurface is the product of an $(n - r)$-dimensional linear subspace and a properly $(r - 1)$-dimensional quadric hypersurface.

Now let M be a hypersurface in E^{n+1}. We consider a parametrization

$$x(u_1, \cdots, u_n) = (u_1, \cdots, u_n, v) \tag{3.2}$$

where $v = v(u_1, \cdots, u_n)$.

Denote $\partial v / \partial u_i$ by v_i. Then we have ([2])

$$g_{ij} = \delta_{ij} + v_i v_j, \quad g^{ij} = \delta_{ij} - v_i v_j / g \tag{3.3}$$

where

$$g = \det(g_{ij}) = 1 + \sum_{i=1}^{n} v_i^2, \tag{3.4}$$

and $g_{ij} =< \partial_i x, \partial_j x >$. The Laplacian Δ of M is given by

$$\Delta = -\sum_{i,j} \Big(\frac{\partial_i g}{2g} g^{ij} + \partial_i g^{ij} \Big) \partial_j - \sum_{i,j} g^{ij} \partial_i \partial_j. \tag{3.5}$$

And the Gauss map G of M is given by

$$G = (G_1, \cdots, G_n, G_{n+1}) = g^{-\frac{1}{2}}(-v_1, \cdots, -v_n, 1). \tag{3.6}$$

If M is a properly n-dimensional quadric hypersurface, then M is one of the following three kinds:

(I) $v = \frac{1}{2} \sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0,$

(II) $v^2 = \sum_{i=1}^{n} a_i u_i^2 + c, \quad a_1 \cdots a_n c \neq 0,$

(III) $v^2 = \sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0.$

4. Proper quadric hypersurfaces of kind (I)

We consider the following parametrization:

$$x = (u_1, \cdots, u_n, v), \quad v = \frac{1}{2}\sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0. \tag{4.1}$$

In this case, we have

$$g_{ij} = \delta_{ij} + a_i a_j u_i u_j, \quad g^{ij} = \delta_{ij} - g^{-1} a_i a_j u_i u_j, \tag{4.2}$$

$$g = \det(g_{ij}) = 1 + \sum_i a_i^2 u_i^2, \tag{4.3}$$

$$\Delta = -g^{-2}\sum_i a_i^3 u_i^2 \sum_j a_j u_j \partial_j + g^{-1}\sum_i a_i \sum_j a_j u_j \partial_j - \sum_{i,j} g^{ij}\partial_i\partial_j \tag{4.4}$$

and we have

$$G = (G_1, \cdots, G_n, G_{n+1}) = g^{-\frac{1}{2}}(-a_1 u_1, \cdots, -a_n u_n, 1). \tag{4.5}$$

LEMMA 2. *For each $k = 1, \cdots, n$ we have*

$$\begin{aligned} &\Delta G_k \\ &= -a_k u_k g^{-\frac{7}{2}}\Big\{4\Big(\sum_i a_i^3 u_i^2\Big)^2 - 2g\sum_i a_i^4 u_i^2 - g\Big(\sum_i a_i\Big)\sum_j a_j^3 u_j^2 \\ &\qquad - 3g a_k \sum_i a_i^3 u_i^2 + g^2 \sum_i a_i^2 + g^2 a_k \sum_i a_i\Big\}. \end{aligned} \tag{4.6}$$

And we have

$$\begin{aligned} &\Delta G_{n+1} \\ &= g^{-\frac{7}{2}}\Big\{4\Big(\sum_i a_i^3 u_i^2\Big)^2 - g\sum_i a_i \sum_j a_j^3 u_j^2 - 2g\sum_i a_i^4 u_i^2 + g^2\sum_i a_i^2\Big\}. \end{aligned} \tag{4.7}$$

Proof. Note that the Gauss map $G = (G_1, \cdots, G_n, G_{n+1})$ is given by $G_k = -a_k u_k g^{-\frac{1}{2}}$ for $1 \leq k \leq n$ and $G_{n+1} = g^{-\frac{1}{2}}$. From (4.2) and

(4.4) we may derive the above formula (4.6) and (4.7) by a straightforward computation.

Now suppose that M satisfies the condition (1.4) with $A = (a_{ij})$, $1 \leq i, j \leq n+1$. Then for each $k = 1, \cdots, n$ we have from (4.6) and (4.7)

$$g^3\left\{\sum_j a_{kj} a_j u_j - a_{k\,n+1}\right\} = a_k u_k \{*\} \tag{4.8}$$

$$g^3\left\{-\sum_j a_{n+1j} a_j u_j + a_{n+1n+1}\right\} = \{**\} \tag{4.9}$$

where $\{*\}$ and $\{**\}$ are the parentheses in the right side of (5.6) and (5.7), respectively. Note that g is a polynomial in $u_1, \cdots, u_n$ of degree 2 and note that the left side of (4.8) is a polynomial in $u_1, \cdots, u_n$ of possible degree 0, 6 or 7 and the right side of (4.8) is a polynomial in $u_1, \cdots, u_n$ of degree less than or equal to 5. Hence we have

$$a_{k\ell} = 0, \quad 1 \leq k \leq n, \quad 1 \leq \ell \leq n+1. \tag{4.10}$$

Similarly from (4.9) we have

$$a_{n+1\,\ell} = 0, \quad 1 \leq \ell \leq n+1. \tag{4.11}$$

Thus (1.4), (4.10) and (4.11) show that M satifies the condition $\Delta G = 0$. Hence by Lemma 1, we see that M is a hyperplane, which is not a quadric hypersurface of kind (I).

5. Proper quadric hypersurfaces of kind (II)

For each hypersurfaces we consider a parametrization

$$x = (u_1, \cdots, u_n, v), \quad v^2 = a_1 u_1^2 + \cdots + a_n u_n^2 + c, \quad a_1 \cdots a_n c \neq 0. \tag{5.1}$$

In this case, we have

$$g_{ij} = \delta_{ij} + W^{-1} a_i a_j u_i u_j, \quad g^{ij} = \delta_{ij} - \tilde{g}^{-1} a_i a_j u_i u_j, \tag{5.2}$$

$$g = 1 + W^{-1} \sum_i a_i^2 u_i^2, \quad g^{-1} = 1 - \tilde{g}^{-1} \sum_i a_i^2 u_i^2, \tag{5.3}$$

where

$$W = v^2 = \sum_i a_i u_i^2, \quad \tilde{g} = gW = c + \sum_i a_i(1+a_i)u_i^2. \tag{5.4}$$

And the Gauss map G of M is given by

$$G = (G_1, \cdots, G_n, G_{n+1}) = \tilde{g}^{-\frac{1}{2}}(-a_1u_1, \cdots, -a_nu_n, v). \tag{5.5}$$

As in Section 4, by a straightforward computation, we have the following:

LEMMA 3. *For each $k = 1, \cdots, n$ we have*

$$\begin{aligned}
&\Delta G_k \qquad\qquad (5.6)\\
=&a_k u_k \tilde{g}^{-\frac{7}{2}} W^{-1}\Big\{2\tilde{g}WB - WCD - \tilde{g}CE + CE^2 - a_k^2\tilde{g}^2W \\
&+ a_k\tilde{g}WD + a_k\tilde{g}^2E - a_k\tilde{g}E^2 + \tilde{g}W\sum_j \alpha_j^2 a_j^2(1+a_j)u_j^2 \\
&- a_k\tilde{g}^2W\alpha_k + 3\tilde{g}WF - \tilde{g}^2W\sum_i a_i(1+a_i) \\
&- \tilde{g}^2Wa_k(1+a_k) - 3WC^2 + 2a_k\tilde{g}WC\Big\}.
\end{aligned}$$

And we have

$$\begin{aligned}
&\Delta G_{n+1} \qquad\qquad (5.7)\\
=&\tilde{g}^{-\frac{7}{2}}W^{-\frac{3}{2}}\Big\{-2\tilde{g}W^2B + W^2CD - \tilde{g}WCE - WCE^2 \\
&+ 2\tilde{g}^2WD - \tilde{g}WED - 2\tilde{g}^2E^2 + \tilde{g}E^3 - \tilde{g}W^2\sum_j \alpha_i a_i^2(1+a_i)u_i^2 \\
&+ \tilde{g}^2W\sum_i \alpha_i a_i^2 u_i^2 + 3W^2C^2 - 3\tilde{g}W^2F + \tilde{g}^2W^2\sum_i a_i(1+a_i) \\
&+ 2\tilde{g}^2WC + \tilde{g}^3E - \tilde{g}^3W\sum_i a_i\Big\},
\end{aligned}$$

where

$$B = \sum_i a_i^3(1+a_i)u_i^2, \quad C = \sum_i a_i^2(1+a_i)u_i^2, \quad D = \sum_i a_i^3u_i^2, \tag{5.8}$$

$$E = \sum_i a_i^2u_i^2, \qquad F = \sum_i a_i^2(1+a_i)u_i^2, \quad \alpha_i = \sum_{j\neq i} a_j.$$

Now suppose that M satisfies the condition (1.4) with $A = (a_{ij})$, $1 \leq i, j \leq n+1$. Then we obtain from (5.6) and (5.7)

$$W\tilde{g}^3 \left\{ a_{k\ n+1} W^{\frac{1}{2}} - \sum_{\ell=1}^{n} a_{k\ell} a_\ell u_\ell \right\} = a_k u_k \{*\},\ k = 1, \cdots, n, \quad (5.9)$$

$$\tilde{g}^3 \left\{ W^{\frac{3}{2}} (-\sum_{\ell=1}^{n} a_{n+1\ \ell} a_\ell u_\ell) + a_{n+1\ n+1} W^2 \right\} = \{**\}, \quad (5.10)$$

where $\{*\}$ and $\{**\}$ are the parentheses in the right side of (5.6) and (5.7), respectively.

From (5.9) we see that $a_{k\ n+1} = 0$ for all $k = 1, \cdots, n$ and that if $a_{k\ell} \neq 0$ for some $1 \leq k, \ell \leq n$ then $\tilde{g}$ must be a constant, that is, $a_i = -1$ for all $i = 1, \cdots, n$. And from (5.10) we see that $a_{n+1\ell} = 0$ for all $\ell = 1, \cdots, n$ and that if $a_{n+1n+1} \neq 0$ then $\tilde{g}$ must be a constant, that is, $a_i = -1$ for all $i = 1, \cdots, n$.

Hence if A is not a zero matrix, then M is a sphere. And if $A = 0$, then by Lemma 1, M is a hyperplane, which is not a quadric hypersurface of kind (II).

6. Proper quadric hypersurfaces of kind (III)

For such hypersurfaces we consider a parametrization

$$x = (u_1, \cdots, u_n, v),\ v^2 = a_1 u_1^2 + \cdots + a_n u_n^2,\ a_1 \cdots a_n \neq 0. \quad (6.1)$$

In Section 5 with $c = 0$, the nondegeneracy of M implies that $\tilde{g} = \sum_{i=1}^{n} a_i(1 + a_i) u_i^2$ is a polynomial of degree 2, or equivalently, $a_i \neq -1$ for some $i = 1, \cdots, n$. And the formulae (5.9) and (5.10) are also valid with $c = 0$.

We now suppose that M satisfies the condition (1.4). As in Section 5, we see that if $A \neq 0$, then we have $a_i = -1$, $i = 1, \cdots, n$, which is a contradiction. And we see that if $A = 0$, then by Lemma 1, M is a hyperplane, which is not a quadric hypersurface of kind (III).

7. Proof of theorem

Suppose that a quadric hypersurface M satisfies the condition (1.4) and that M is not a hyperplane. If M is a quadric cylindrical hypersurface in E^{n+1}, then M is the product of a proper quadric hypersurface N^p in E^{p+1} and a linear subspace E^{n-p}. Since N aslo satisfies the condition (1.4) with a suitable square matrix, N is a hypersphere $S^p(r)$ in E^{p+1}. This completes the proof of the theorem.

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