J. Korean Math. Soc. **31** (1994), No. 3, pp. 375–391

SEQUENTIAL OPERATOR–VALUED FUNCTION SPACE INTEGRAL AS AN $\mathcal{L}(L_p, L_{p'})$ THEORY

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1. Introduction and Preliminaries

In 1968, Cameron and Storvick introduced the analytic and the sequential operator-valued function space integral [2]. Since then, the theory of the analytic operator-valued function space integral has been investigated by many mathematicians - Cameron, Storvick, Johnson, Skoug, Lapidus, Chang and author etc. But there are not that many papers related to the theory of the sequential operator-valued function space integral. In this paper, we establish the existence of the sequential operator-valued function space integral as an operator from L_p to $L_{p'}(1 < p < 2)$ and investigated the integral equation related to this integral.

Now, we present some necessary notations and some facts which are needed in our subsequential sections. Insofar as possible, we adopt the definitions and notations of [2] and [9].

A. Let $\mathbf{C}$, $\mathbf{C}_+$ and $\mathbf{C}_+^\sim$ be the set of all complex numbers, all complex numbers with positive real part and non-zero complex numbers with non-negative real part, respectively. Given a real numbers d such that $1 \le d \le +\infty$, d and d' will be always related by $\frac{1}{d} + \frac{1}{d'} = 1$. If $1 < p < 2$ is given, let α in $(1, +\infty)$ be such that $\alpha = \frac{p}{2-p}$. And N will be a natural number restricted so that $N < 2\alpha$, r will be a real number such that $\frac{2\alpha}{2\alpha - N} < r < +\infty$ and we let $\delta \equiv \frac{N}{2\alpha}$. Note that $0 < r'\delta < 1$.

Received April 19, 1993. Revised September 2, 1993.

1980 Mathematics Subject Classification. Primary 28C20; Secondary 28A 33.

Key words and phrases. Bochner integral, Wiener integral, analytic operator-valued functionspace integral, sequential operator-valued function space integral, ope rator-valued Feynman integral.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation,1992.

B. Let $1 < p < 2$ be given. For λ in $\mathbf{C}_+^\sim$, ψ in $L_p(\mathbf{R}^N)$, ξ in $\mathbf{R}^N$ and a positive real s, let

$$(C_{\lambda/s}\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{N/2} \int_{\mathbf{R}^N} \psi(u) \exp\left(-\frac{\lambda\|u-\xi\|^2}{2s}\right) dm_L(u) \qquad (1.1)$$

where m_L is the Lebesgue measure on $\mathbf{R}^N$, if N is odd, we always choose $\lambda^{-\frac{1}{2}}$ with non-negative real part and if $Re\lambda = 0$ the integral in the above should be interpreted in the mean just as in the theory of the L_p Fourier transform. From [9], $C_{\lambda/s}$ is in $\mathcal{L}(L_p, L_{p'})$, the space of all bounded linear operator from $L_p(\mathbf{R}^N)$ to $L_{p'}(\mathbf{R}^N)$, $\|C_{\lambda/s}\| \leq (|\lambda|/2\pi s)^\delta$ and as a functions of λ, $C_{\lambda/s}$ is analytic in $\mathbf{C}_+$ and strongly continuous in $\mathbf{C}_+^\sim$.

REMARK. In the above notations, for λ in $\mathbf{C}_+$ and ϕ in $L_{p'}(\mathbf{R}^N)$, by the Hölder inequality, we have $|(C_{\lambda/s}\phi)(\xi)| \leq (|\lambda|/pRe\lambda)^{N/2p}(\frac{|\lambda|}{2\pi s})^{N/2} \times(\frac{2\pi s}{pRe\lambda})^{N/2p}\|\phi\|_{p'}$. Hence, for all ξ in $\mathbf{R}^N$, $(C_{\lambda/s}\phi)(\xi)$ is well-defined. Therefore, from the Fubini theorem and the Chapman-Kolmogorov equation [6], $C_{\lambda/s_1}(C_{\lambda/s_2}\psi) = C_{\lambda/(s_1+s_2)}\psi$ where s_1 and s_2 are positive real numbers and ψ is in $L_p(\mathbf{R}^N)$. Hence, we use the convention

$$C_{\lambda/s_1}(C_{\lambda/s_2}\psi) = (C_{\lambda/s_1} \circ C_{\lambda/s_2})\psi. \qquad (1.2)$$

C. Let $t > 0$ be given. $M^\sim[0,t]$ will denote the set of all complex Borel measures η on the interval $[0,t]$ which satisfies the following conditions;

(1) $\eta(\{0\}) = \eta(\{t\}) = 0$
(2) If μ is the continuous part of η, then the Radon-Nikodym derivative $d|\mu|/dm$ exists and is essentially bounded where m is ordinary Lebesgue measure on $[0,t]$.

D. Let $C[0,t] \equiv C$ be the space of $\mathbf{R}^N$-valued continuous functions y on $[0,t]$. Let $C_0[0,t] \equiv C_0$ be the subspace of C which vanishes at 0. We consider as equipped with N-dimensional Wiener measure m_w. Infact, every element y in C has a unique reprsentation $y = x + \xi$ where

x is in C_0 and ξ in $\mathbf{R}^N$. Let $\mathcal{S}[0,t] \equiv \mathcal{S}$ denote the space of all piecewise continuous functions on $[0,t]$. In this paper, we will be concerned with only the uniform topologies on C, C_0 and $\mathcal{S}$, respectively.

E. For η in $M^{\sim}[0,t]$, let $L_{\alpha r:\eta}{}^{\sim}([0,t] \times \mathbf{R}^N) \equiv L^{\sim}_{\alpha r:\eta}$ be the space of all $\mathbf{C}$-valued functions θ such that θ is continuous $\eta \times m_L - a.e.$ on $[0,t] \times \mathbf{R}^N$ and

$$\|\theta\|_{\alpha r:\eta} \equiv \left\{ \int_{[0,t]} \|\theta(s,\cdot)\|_\alpha^r \, d|\eta|(s) \right\}^{\frac{1}{r}} \tag{1.3}$$

and

$$\|\theta\|_{\infty 1:\eta} \equiv \int_{[0,t]} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s) \tag{1.4}$$

are finite.

For θ in $L_{\alpha r:\eta}{}^{\sim}$, a measurable subset A of $[0,t]$, we define a function $\theta(A)$ on $\mathbf{R}^N$ given by

$$[\theta(A)](u) = \int_A \theta(s,u)\, d\eta(s). \tag{1.5}$$

By the Minkowski's integral inequality [11], the Hölder inequality and the Fubini theorem, we have $\theta(A)$ is in $L_\alpha(\mathbf{R}^N) \cap L_\infty(\mathbf{R}^N)$ for any measurable subset A of $[0,t]$.

REMARK. Let $1 \le q < +\infty$ be given and let H be a $\mathbf{C}$- valued measurable function on $[0,t] \times \mathbf{R}^N$ such that $H(s,\cdot)$ is in $L_q(\mathbf{R}^N)$ for $\eta-$ $a.e.$ s and $\int_{[0,t]} \|H(s,\cdot)\|_q \, d|\eta|(s)$ is finite. By the Pettis's measurability theorem, the Minkowski's integral inequality and **Lemma 1** in [4, p52], we can prove that $H(s,\cdot)$ is Bochner integrable with respect to η and

$$B - \int_{[0,t]} H(s,\cdot)\, d\eta(s) = L - \int_{[0,t]} H(s,\cdot)\, d\eta(s) \tag{1.6}$$

for $m_L - a.e.$ where $B - (L-) \int_{[0,t]} H(s,\cdot)\, d\eta(s)$ refers to the Bochner (Lebesgue) integral, respectively. In general, the equality (1.6) is not true whenever $q = \infty$.

Let ϕ be in $L_\alpha(\mathbf{R}^N)$. From **Lemma 1.3** in [9, p129], a function $M_\phi : L_{p'}(\mathbf{R}^N) \longrightarrow L_p(\mathbf{R}^N)$ defined by $M_\phi(f) = f\phi$, is in $\mathcal{L}(L_{p'}, L_p)$ and $\|M_\phi\| \le \|\phi\|_\alpha$. It will be convenient let $\theta(s)$ denote $M_{\theta(s,\cdot)}$ and θ_A denote $M_{\theta(A)(\cdot)}$ for θ in $L^{\sim}_{\alpha r:\eta}$ where A is a measurable subset of $[0,t]$.

F. Let F be a functional on C. Given $\lambda > 0$, ψ in $L_p(\mathbf{R}^N)$ and ξ in $\mathbf{R}^N$, let

$$[I_\lambda(F)\psi](\xi) = \int_{C_0} F(\lambda^{-1/2}x + \xi)\,\psi(\lambda^{-1/2}x(t) + \xi)\,dm_w(x). \tag{1.7}$$

If for $m_L - a.e.$ ξ in $\mathbf{R}^N$, $[I_\lambda(F)\psi](\xi)$ exists in $L_{p'}(\mathbf{R}^N)$ and if the correspondence $\psi \longrightarrow I_\lambda(F)\psi$ gives an element of $\mathcal{L}(L_p, L_{p'})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ. Suppose there exists λ_0, λ_1 $(0 < \lambda_0 < \lambda_1 \le +\infty)$ such that $I_\lambda(F)$ exists for all $\lambda_0 < \lambda < \lambda_1$ and there exists an $\mathcal{L}(L_p, L_{p'})$-valued function which is analytic in $\mathbf{C}_{+,\lambda_0,\lambda_1} \equiv \mathbf{C}_+ \cap \{z \in \mathbf{C} \mid \lambda_0 < |\lambda| < \lambda_1\}$ and agree with $I_\lambda(F)$ on (λ_0, λ_1), then this function is called the operator-valued function space integral of F associated with λ_0 and in this case we say that $I^{an}_\lambda(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0,\lambda_1}$. If $I^{an}_\lambda(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0,\lambda_1}$, $I_\lambda(F)$ is strongly continuous in $\mathbf{C}^{\sim}_{+,\lambda_0,\lambda_1} \equiv \mathbf{C}^{\sim}_{+} \cap \{z \in \mathbf{C} \mid \lambda_0 < |\lambda| < \lambda_1\}$ we say that $I^{an}_\lambda(F)$ exists for λ in $\mathbf{C}^{\sim}_{+,\lambda_0,\lambda_1}$. When λ is purely imaginary, $I^{an}_\lambda(F)$ is called the analytic operator-valued Feynman integral of F.

G. Let $\sigma : 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ be any partition of $[0,t]$ and the norm of σ, denoted by $\|\sigma\|$, equals $\max_{i=1,2,\ldots,n}(t_i - t_{i-1})$.

For x in C, we define

$$x_\sigma(s) = \begin{cases} x(t_{i-1}), & \text{if } t_{i-1} \le s < t_i \\ x(t), & \text{if } s = t. \end{cases} \tag{1.8}$$

If σ is given as in the above and $\{v_0, v_1, \cdots, v_n\}$ is any set of $n+1$ elements of $\mathbf{R}^N$, we define a function

$$z(\sigma; v_0, v_1, \cdots, v_n; s) \equiv \begin{cases} v_{i-1}, & \text{if } t_{i-1} \le t < t_i \\ v_n, & \text{if } s = t. \end{cases} \tag{1.9}$$

Clearly, x_σ and $z(\sigma; v_0, v_1, \cdots, v_n; \cdot)$ are in $\mathcal{S}$. And, for a given functional F on $\mathcal{S}$, $f_\sigma(v_0, v_1, \cdots, v_n)$ is given by

$$f_\sigma(x(t_0), x(t_1), \cdots, x(t_n)) \equiv F(z(\sigma; x(t_0), \cdots, x(t_n); \cdot)) \tag{1.10}$$

for x in C.

Let λ be in $\mathbf{C}_+$ and let ψ be in $L_p(\mathbf{R}^N)$. Let F be a $\mathbf{C}$-valued functional on $\mathcal{S}$. For a given partition $\sigma : 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the operator $I_\lambda^\sigma(F)$ is defined by the formula

$$\begin{aligned} &[I_\lambda^\sigma(F)\psi](\xi) \\ =&\lambda^{nN/2}\{(2\pi)^n t_1(t_2 - t_1)\cdots(t_n - t_{n-1})\}^{-N/2} \\ &\times \int_{\mathbf{R}^{nN}} f_\sigma(v_0, v_1, \cdots, v_n)\psi(v_n) \\ &\times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{\|v_j - v_{j-1}\|^2}{t_j - t_{j-1}}\right\} d\prod_{i=1}^{n} m_L(v_i) \end{aligned} \tag{1.11}$$

where $v_0 = \xi$.

Suppose there exists $\lambda_0, \lambda_1 (0 < \lambda_0 < \lambda_1 \leq +\infty)$ such that for all λ in $\mathbf{C}_{+,\lambda_0,\lambda_1}$, $I_\lambda^\sigma(F)\psi$ exists for all ψ in $L_p(\mathbf{R}^N)$ and the weak limit $I_\lambda^{seq}(F) \equiv \omega - \lim_{\|\sigma\|\to 0} I_\lambda^\sigma(F)$ exists in $\mathcal{L}(L_p, L_{p'})$, we say that $I_\lambda^{seq}(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0,\lambda_1}$. If $I_\lambda^{seq}(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0,\lambda_1}$ and the strongly limit $I_{-iq}^{seq}(F) \equiv s - \lim_{\substack{\lambda \in \mathbf{C}_{+,\lambda_0,\lambda_1} \\ \lambda \to -iq}} I_\lambda^{seq}(F)$ exists in $\mathcal{L}(L_p, L_{p'})$ for $\lambda_0 < |q| < \lambda_1$, we say that $I_{-iq}^{seq}(F)$ exists in $\mathbf{C}_{+,\lambda_0,\lambda_1}^{\sim}$, this is called the sequential operator-valued Feynman integral of F. When $I_\lambda^{an}(F)$ and $I_\lambda^{seq}(F)$ exists in $\mathbf{C}_{+,\lambda_0,\lambda_1}^{\sim}$ for some $\lambda_0 < \lambda_1$ and these are same, we say that $I_\lambda^{f}(F) = I_\lambda^{an}(F) = I_\lambda^{seq}(F)$ is the operator-valued Feynman integral of F.

Let θ be in $L_{\alpha r;\eta}^{\sim}$ and let $\sigma : 0 = t < t_1 < \cdots < t_n = t$ be a given partition on $[0, t]$. Let

$$F(y) = \int_{[0,t]} \theta(s, y(s))\, d\eta(s) \qquad \text{for } y \text{ in } \mathcal{S} \tag{1.12}$$

for which the integral exists. From **Lemma 0.1** of [8] and the elementary calculus, we have, for $\lambda > 0$, $F(\lambda^{-1/2}x + \xi)$ and $F(\lambda^{-1/2}x_\sigma + \xi)$ is defined for $m_w \times m_L - a.e.\ (x, \xi)$ in $C_0 \times \mathbf{R}^N$ and

$$|F(\lambda^{-1/2}x + \xi)| \leq \|\theta\|_{\infty 1;\eta}. \tag{1.13}$$

H. Let X and Y be two Banach spaces, $\mathcal{L}(X,Y)$ a space of bounded linear, operators from X into Y and (Ω, m) be a measure space. Let $G : \Omega \longrightarrow Z(X,Y)$ be a function such that for each x in X, $\{G(s)\}(x)$ is Bochner integrable with respect to m. Then there exists a linear operator J from X to Y such that

$$J(x) = B - \int_{\Omega} \{G(s)\}(x)\, dm(s) \qquad \text{for } x \text{ in } X. \tag{1.14}$$

This linear operator J is denoted by $(BS) - \int_{\Omega} \{G(s)\}\, dm(s)$. When $X = Y$, J is called the strongly integral of G [see 3].

2. Sequential operator-valued function space integral

In 1992, Chang and the author introduced the theory of the analytic operator -valued function space integral as an operator L_p to $L_{p'}$ $(1 < p < 2)$ for certain functional involving some Borel measures [2]. In this section, we will establish the existence of the sequential operator-valued function space integral as an operator L_p to $L_{p'}$ under the some conditions different from the conditions given in [2].

Throughout this section, let $1 < p < 2$ be given and let η be in $M^{\sim}[0,t]$. Assume μ is a continuous part of η and $\nu = \sum_{p=1}^{h} \omega_p \delta_{\tau_p}$ is a discret part of η where δ_τ is the Dirac measure with total mass one concentrated at τ and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_h < \tau_{h+1} = t$. And let j be a positive integer such that $\Gamma(j(1-r'\delta))$ is a minimum value of $\{\Gamma(i(1-r'\delta)) \,|\, i \text{ is a natural number}\}$ where Γ is a Gamma function. For λ in $\mathbf{C}_+^{\sim}$, let

$$B(\lambda) = \left(\frac{|\lambda|}{2\pi}\right)^{(h+1)\delta} \Gamma(j(1-r'\delta))^{-(h+1)/r'} \times \left\{\prod_{p=1}^{h+1} (\tau_p - \tau_{p-1})^{-\delta}\right\} \Gamma(1-r'\delta)^{(h+1)/r'} \quad \text{and} \tag{2.1}$$

$$K = \sum_{p=1}^{h+1} (\tau_p - \tau_{p-1})^{1-r'\delta}.$$

And for non-negative integers $q_0, q_1, \cdots, q_h$ and $j_1, j_2, \cdots, j_{h+1}$ with $j_1 + j_2 + \cdots + j_{h+1} = q_0$, let

$$\begin{aligned}&\Delta_{q_0; j_1, j_2, \cdots, j_{h+1}} \qquad (2.2)\\ =&\{(s_1, s_2, \cdots, s_{q_0}) \text{ in } [0,t]^{q_0} \mid 0 < s_1 < s_2 < \cdots < s_{j_1} < \tau_1 < s_{j_1+1} < \\ &\cdots < s_{j_1+j_2} < \tau_2 < \cdots < \tau_h < s_{j_1+\cdots+j_h+1} < \\ &\cdots < s_{j_1+\cdots+j_{h+1}} = s_{q_0} < t\}\end{aligned}$$

and for $(s_1, s_2, \cdots, s_{q_0})$ in $\Delta_{q_0; j_1, j_2, \cdots, j_{h+1}}$, m in $\{1, 2, \cdots, h\}$, θ in $L^{\sim}_{\alpha r:\eta}$ and λ in $\mathbf{C}^{\sim}_{+}$, let

$$\begin{aligned}&L_m \qquad (2.3)\\ =&\theta(\tau_m)^{q_m} \circ C_{\lambda/(s_{j_1+\cdots+j_m+1} - \tau_m)} \circ \theta(s_{j_1+\cdots+j_m+1}) \\ &\circ C_{\lambda/(s_{j_1+\cdots+j_m+2} - s_{j_1+\cdots+j_m+1})} \circ \cdots \\ &\circ \theta(s_{j_1+j_2+\cdots+j_{m+1}}) \circ C_{\lambda/(\tau_{m+1} - s_{j_1+\cdots+j_{m+1}})}.\end{aligned}$$

Hence, we let $\theta(\tau)^{\circ} \equiv 1$, an identity map on $L_{p'}(\mathbf{R}^N)$. For a natural number m, let

$$F^m(y) = \left\{ \int_{[0,t]} \theta(s, y(s))\, d\eta(s) \right\}^m \quad \text{for } y \text{ in } \mathcal{S}. \qquad (2.4)$$

If $m = 0$, from the definition, directly $I^{an}_{\lambda}(F^{\circ}) = C_{\lambda/t}$ for λ in $\mathbf{C}^{\sim}_{+}$.

THEOREM 2.1. *Under the assumptions and notations in the above, the operator $I^{an}_{\lambda}(F^m)$ exists for λ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$ and for all λ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$,*

$$\begin{aligned}I^{an}_{\lambda}(F^m) =& m! \sum_{q_0+\cdots q_h = m} \frac{w_1^{q_1} \cdots w_h^{q_h}}{q_1! \cdots q_h!} \sum_{j_1+\cdots+j_{h+1} = q_0} (BS) \qquad (2.5)\\ &- \int_{\Delta_{q_0; j_1, \cdots j_1, \cdots, j_{h+1}}} L_0 \circ L_1 \circ \cdots \circ L_h\, d \prod_{i=1}^{q_0} \mu(s_i).\end{aligned}$$

Moreover, for all λ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$,

$$\begin{aligned}&\|I^{an}_{\lambda}(F^m)\| \qquad (2.6)\\ \le&(m!)^{1/r'} B(\lambda) \left\{ \left\| \frac{d \mid \mu \mid}{dm} \right\|_{\infty}^{1/r'} \Gamma(1 - r'\delta)^{1/r'} \left(\frac{\mid \lambda \mid}{2\pi} \right)^{\delta} K^{(1-r'\delta)/r'} \|\theta\|_{\alpha r:\mu} \right.\\ &\left. + \sum_{l=1}^{h} \mid w_l \mid \left(\|\theta(\tau_l, \cdot)\|_{\infty} \vee \|\theta(\tau_l, \cdot)\|_{\alpha} \right) \right\}^m.\end{aligned}$$

Proof. From the similar method as in the proof of **Theorem 2.3** in [2,p704], the operator $I_\lambda^{an}(F^m)$ exists for all λ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$ and the equality (2.5) holds. Now, let λ be in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$ and let ψ be in $L_p(\mathbf{R}^N)$. Then

$$\|I_\lambda^{an}(F^m)\psi\|_{p'} \tag{2.7}$$

$$\overset{(1)}{\leq} m!\|\psi\|_p \sum_{q_0+\cdots+q_h=m} \frac{|w_1|^{q_1}\cdots|w_h|^{q_h}}{q_1!\cdots q_h!}\times$$

$$\sum_{j_1+\cdots+j_{h+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}} \|L_0\circ L_1\circ\cdots\circ L_h\|\, d\prod_{i=1}^{q_0}|\mu|(s_i)$$

$$\overset{(2)}{\leq} m!\|\psi\|_p \sum_{q_0+\cdots q_h=m} \frac{|w_1|^{q_1}\cdots|w_h|^{q_h}}{q_1!\cdots q_h!}\left(\frac{|\lambda|}{2\pi}\right)^{(q_0+h+1)\delta}$$

$$\times\prod_{l=1}^{h}(\|\theta(\tau_l,\cdot)\|_\infty\vee\|\theta(\tau_l,\cdot)\|_\alpha)^{q_l}$$

$$\times\sum_{j_1+\cdots+j_{h+1}=q_0}\int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}}\left(\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_\alpha\right)$$

$$\times\{(s_1(s_2-s_1)\cdots(\tau-s_{j_1})\cdots(t-s_{q_0})\}^{-\delta} d\prod_{i=1}^{q_0}|\mu|(s_i)$$

$$\overset{(3)}{\leq} m!\|\psi\|_p \sum_{q_0+\cdots+q_h=m} \frac{|w_1|^{q_1}\cdots|w_h|^{q_h}}{q_1!\cdots q_h!}\left(\frac{|\lambda|}{2\pi}\right)^{(q_0+h+1)\delta}$$

$$\times\prod_{l=1}^{h}(\|\theta(\tau_l,\cdot)\|_\infty\vee\|\theta(\tau_l,\cdot)\|_\alpha)^{q_l}$$

$$\times\left\{\int_{\Delta_{q_0}}\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_\alpha^r\, d\prod_{i=1}^{q_0}|\mu|(s_i)\right\}^{1/r}$$

$$\times\left\{\sum_{j_1+\cdots+j_{h+1}=q_0}\int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}}\{(s_1(s_2-s_1)\cdots(t-s_{q_0})\}^{-r'\delta}\right.$$

$$\left.\times\, d\prod_{i=1}^{q_0}|\mu|(s_i)\right\}^{1/r'}$$

$$\overset{(4)}{\leq} m!\|\psi\|_p \times \sum_{q_0+\cdots+q_h=m} \frac{|w_1|^{q_1}\cdots|w_h|^{q_h}}{q_1!\cdots q_h!}\left(\frac{|\lambda|}{2\pi}\right)^{(q_0+h+1)\delta}(q_0!)^{-1/r}$$

$$\times \prod_{l=1}^{h}(\|\theta(\tau_l,\cdot)\|_\infty \vee \|\theta(\tau_l,\cdot)\|_\alpha)^{q_l}\|\theta\|_{\alpha r:\mu}^{q_0}\left\|\frac{d|\mu|}{dm}\right\|_\infty^{q_0/r'}$$

$$\times\left\{\sum_{j_1+\cdots+j_{h+1}=q_0}\frac{\prod_{p=1}^{h+1}\left\{(\tau_p-\tau_{p-1})^{j_p-(j_p+1)r'\delta}\Gamma(1-r'\delta)^{j_p+1}\right\}}{\prod_{p=1}^{h+1}\Gamma((j_p+1)(1-r'\delta))}\right\}^{1/r'}$$

$$\overset{(5)}{\leq}\|\psi\|_p\left(\frac{|\lambda|}{2\pi}\right)^{(h+1)\delta}\Gamma(j(1-r'\delta))^{-(h+1)/r'}$$

$$\times\prod_{p=1}^{h+1}(\tau_p-\tau_{p-1})^{-\delta}\Gamma(1-r'\delta)^{(h+1)/r'}\sum_{q_0+\cdots+q_h=m}\frac{m!}{q_0!q_1!\cdots q_h!}(q_0!)^{1/r'}$$

$$\times\left\{\left(\frac{|\lambda|}{2\pi}\right)^{\delta}\left\|\frac{d|\mu|}{dm}\right\|_\infty^{1/r'}\Gamma(1-r'\delta)^{1/r'}K^{(1-r'\delta)}\|\theta\|_{\alpha r:\mu}\right\}^{q_0}$$

$$\times\prod_{l=1}^{h}\left\{|w_l|\left(\|\theta(\tau_l,\cdot)\|_\infty\vee\|\theta(\tau_l,\cdot)\|_\alpha\right)\right\}^{q_l}$$

$$\overset{(6)}{\leq}(m!)^{1/r'}\|\psi\|_p B(\lambda)\left[\left\|\frac{d|\mu|}{dm}\right\|_\infty^{1/r'}\Gamma(1-r'\delta)^{1/r'}\left(\frac{|\lambda|}{2\pi}\right)^{\delta}K^{(1-r'\delta)/r'}\right.$$

$$\left.\times\|\theta\|_{\alpha r:\mu}+\sum_{l=1}^{h}\left\{|w_l|\left(\|\theta(\tau_l,\cdot)\|_\infty\vee\|\theta(\tau_l,\cdot)\|_\alpha\right)\right\}\right]^m.$$

Step (1) results from the properties of the Bochner integral and the multinomial expansion. For any non-negative integer m, $\|\theta(\tau)^m\| \leq (\|\theta(\tau,\cdot)\|_\infty \vee \|\theta(\tau,\cdot)\|_\alpha)^m$ and forλ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$, $\|C_{\lambda/s_1}\circ C_{\lambda/s_2}\| \leq (|\lambda|/2\pi(s_1+s_2))^\delta \leq (|\lambda|/2\pi s_1)^\delta(|\lambda|/2\pi s_2)^\delta$. Hence we have Step (2). Using the same techniques as in the proof of **Theorem 2.2** in [2, p703] and the Hölder inequality, we obtain Step (3). Since $\int_a^b\int_a^{s_m}\cdots\int_a^{s_2}\{(s_1-a)(s_2-s_1)\cdots(b-s_m)\}^{-u}\,ds_1\,ds_2\cdots ds_m = \frac{(b-a)^{m-(m+1)u}\Gamma(1-u)^{m+1}}{\Gamma((m+1)(1-u))}$ for $0<u<1$,we have Step (4). Step (5) follows from the some basic inequalities. From the notation (2.1) and multinomial expression, we have Step (6). Therefore, the theorem is proved.

From the above **Theorem** and the ratio test, we have the following theorem.

THEOREM 2.2. *Under the assumptions in the above, if we let*

$$F(y) = \exp\left\{\int_{[0,t]} \theta(s, y(s))\, d\eta(s)\right\} \quad \text{for } y \text{ in } \mathcal{S}, \tag{2.8}$$

the operator $I_\lambda^{an}(F)$ *exists for all* λ *in* $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$ and *for all* λ *in* $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$,

$$\begin{aligned} &I_\lambda^{an}(F) \\ &= \sum_{m=0}^{\infty} \sum_{q_0+\cdots q_h=m} \frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!} \sum_{j_1+\cdots+j_{h+1}=q_0} \\ &\quad \times (BS) - \int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}} L_0 \circ L_1 \circ \cdots \circ L_h \, d\prod_{i=1}^{q_0} \mu(s_i). \end{aligned} \tag{2.9}$$

Moreover, for λ *in* $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$,

$$\|I_\lambda^{an}(F)\| \qquad \text{is finite.} \tag{2.10}$$

Now, we will establish the existence of the sequential operator-valued function space integral as an operator L_p to $L_{p'}(1 < p < 2)$ for certain functional which involving some Borel measures. Before giving the existence of this integral, we investigate the some properties of $I_\lambda^\sigma(F)$.

Suppose there is a separable dense subset T of $[0, t]$ such that for t in T,

$$\lim_{s>0} \frac{\|\theta[t, t+s)\|_\infty \vee \|\theta[t, t+s)\|_p}{s^{\delta+1}} \le \frac{1}{t^\delta}.$$

Let $\sigma : 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ be a partion on $[0, t]$ with t_i in T for $i = 1, 2, \cdots, n$, let ψ in $L_p(\mathbf{R}^N)$, let λ be in $\mathbf{C}^{\sim}_+$ and let ξ be in

$\mathbf{R}^N$. Then, by the definition of f_σ and $I_\lambda^\sigma(F)$,

$$f_\sigma^m(x(t_0), x(t_1), \cdots, x(t_n)) = \left(\sum_{i=1}^n \int_{[t_{i-1},t_i)} \theta(s, x(t_{i-1}))\, d\eta(s)\right)^m \tag{2.11}$$

and

$$\begin{aligned}
&[I_\lambda^\sigma(F^m)\psi](\xi) \qquad (2.12)\\
=&\lambda^{nN/2}\Big\{(2\pi)^{nN} t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big\}^{-1/2}\\
&\times \int_{\mathbf{R}^{nN}} \psi(v_n)\left(\sum_{j=1}^n \int_{[t_{j-1},t_j)} \theta(s, v_{j-1})\, d\eta(s)\right)^m\\
&\times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n \frac{\|v_j - v_{j-1}\|^2}{(t_i - t_{j-1})}\right\} d\prod_{i=1}^n m_L(v_i)\\
=&\lambda^{nN/2}\Big\{(2\pi)^{nN} t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big\}^{-1/2} \sum_{k=0}^m \frac{m!}{(m-k)!k!}\\
&\times \left(\int_{[t_0,t_1)} \theta(s,\xi)\, d\eta(s)\right)^{m-k} \sum_{q_1+\cdots+q_{n-1}=k,\, q_i\ge 0} \frac{k!}{q_1!\cdots q_{n-1}!}\\
&\times \int_{\mathbf{R}^{nN}} \prod_{i=1}^{n-1}\left(\int_{[t_i,t_{i+1})} \theta(s, v_i)\, d\eta(s)\right)^{q_i} \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n \frac{\|v_j - v_{j-1}\|^2}{(t_j - t_{j-1})}\right\}\\
&\times \psi(v_n)\, d\prod_{i=1}^n m_L(v_i)\\
=&\sum_{k=0}^m \frac{m!}{(m-k)!k!}\theta_{[0,t_1)}^{m-k} \sum_{q_1+\cdots+q_{n-1}=k} \frac{k!}{q_1!\cdots q_{n-1}!}\\
&\times \left[C_{\lambda/t_1} \circ \theta_{[t_1,t_2)}^{q_1} \circ C_{\lambda/(t_2-t_1)} \circ \cdots \theta_{[t_{n-1},t_n)}^{q_{n-1}} \circ C_{\lambda/(t_n-t_{n-1})}\psi\right](\xi)
\end{aligned}$$

THEOREM 2.3. *Under the assumptions and notations in the above, $I_\lambda^\sigma(F^m)$ is well-defined as an operator from L_p to $L_{p'}$ for all λ in $\mathbf{C}_+^\sim$,*

$I_\lambda^\sigma(F^m)$ *is analytic in* $\mathbf{C}_+$ *and is strongly continuous in* $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$. *And*

$$\|I_\lambda^\sigma(F^m)\| \quad \textit{is finite.} \tag{2.13}$$

Proof. From the equality (2.12) in the above, clearly $I_\lambda^\sigma(F^m)$ is well-defined for all λ in $\mathbf{C}_+^{\sim}$. Let ψ and ϕ be in $L_p(\mathbf{R}^N)$. Then

$$\begin{aligned}
&\Big(I_\lambda^\sigma(F^m)\psi,\phi\Big) \qquad (2.14)\\
=&\lambda^{nN/2}\Big\{(2\pi)^{nN}t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big\}^{-1/2}\\
&\times\int_{\mathbf{R}^{(n+1)N}}\psi(v_n)\phi(v_0)\Big(\sum_{j=1}^n\int_{[t_{j-1},t_j)}\theta(s,v_{j-1})\,d\eta(s)\Big)^m\\
&\times\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^n\frac{\|v_j-v_{j-1}\|^2}{(t_j-t_{j-1})}\Big\}\,d\prod_{i=0}^n m_L(v_i).
\end{aligned}$$

The integrand of the right side of the equality (2.14) is dominated by $\prod_{i=0}^n m_L$-integrable function

$$\begin{aligned}
&(|\lambda|+1)^{nN/2}\Big\{(2\pi)^{nN}t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big\}^{-1/2}\\
&\times|\varphi(v_n)\,\phi(v_0)|\,\|\theta\|^m_{\infty 1;\eta}\exp\Big\{-\frac{Re\,\lambda}{2}\sum_{j=1}^n\frac{\|v_j-v_{j-1}\|^2}{t_j-t_{j-1}}\Big\}.
\end{aligned} \tag{2.15}$$

Hence, by the Dominated Convergence Theorem, $(I_\lambda^\sigma(F^m)\psi,\phi)$ is a continuous function of λ in $\mathbf{C}_+$. Using the Morera's Theorem, we can conclude that $(I_\lambda^\sigma(F^m)\psi,\phi)$ is analytic for λ in $\mathbf{C}_+$ and so $I_\lambda^\sigma(F^m)$ is analytic for λ in $\mathbf{C}_+$ [see 12, p189]. Let q be a given real number with $|q|>2\pi t$. For $1\le l\le n$, let

$$\begin{aligned}
A_l=&\theta^{m-k}_{[0,t_1)}\circ C_{-iq/t_1}\circ\cdots\circ C_{-iq/(t_l-t_{l-1})}\\
&\circ\,\theta^{q_l}_{[t_l,t_{l+1})}\circ C_{\lambda/(t_{l+1}-t_l)}\circ\cdots C_{\lambda/(t_n-t_{n-1})}\psi
\end{aligned} \tag{2.16}$$

and let

$$\psi_l = \theta^{q_l}_{[t_l,t_{l+1})} \circ C_{\lambda/(t_{l+1}-t_l)} \circ \cdots \circ C_{\lambda/(t_n-t_{n-1})}\psi. \tag{2.17}$$

Then

$$\begin{aligned}
&\|I^\sigma_\lambda(F^m)\psi - I^\sigma_{-iq}(F^m)\psi\|_{p'} \qquad (2.18)\\
&\le \sum_{k=0}^{m} \frac{m!}{(m-k)!k!} \sum_{q_1+\cdots+q_{n-1}=k} \frac{k!}{q_1!\cdots q_{n-1}!} \sum_{l=1}^{n} \|A_l - A_{l-1}\|_{p'}\\
&\le \sum_{k=0}^{m} \frac{m!}{(m-k)!k!} \sum_{q_1+\cdots+q_{n-1}=k} \frac{k!}{q_1!\cdots q_{n-1}!}\\
&\times \sum_{l=1}^{n} \prod_{i=0}^{l-1} \left(\|\theta_{(t_{i-1},t_i)}(\cdot)\|_\infty \vee \|\theta_{[t_{i-1},t_i)}\|_\alpha\right)^{q_i}\\
&\times \prod_{j=1}^{l-1} \left\{\frac{|q|}{2\pi(t_j - t_{j-1})}\right\}^{\delta} \left\|C_{\lambda/(t_j-t_{j-1})}\psi_l - C_{-iq/(t_j-t_{j-1})}\psi_l\right\|_{p'}
\end{aligned}$$

where $q_0 = m-k$. From **B** in Section 1, $I^\sigma_\lambda(F^m)$ is strongly continuous on $\mathbf{C}^\sim_{+,2\pi t,+\infty}$. And the inequality (2.13) is clear by the elementary calculus. Hence, the proof of this theorem finished.

THEOREM 2.4. *Under the assumptions and notations in the above, $I^{seq}_\lambda(F^m)$ exists for λ in $\mathbf{C}^\sim_{+,2\pi t,+\infty}$ and $I^{seq}_\lambda(F^m) = I^{an}_\lambda(F^m)$ for λ in $\mathbf{C}^\sim_{+,2\pi t,+\infty}$. Moreover, the operator-valued Feynman integral $I^f_\lambda(F^m)$ exists for λ in $\mathbf{C}^\sim_{+,2\pi t,+\infty}$.*

Proof. Let M be a given positive real number, let $2\pi t < \lambda < M$, let σ be a partition on $[0,t]$ and let ψ, φ be in $L_p(\mathbf{R}^N)$. Then by the Wiener integration formula,

$$\begin{aligned}
&[I^\sigma_\lambda(F^m)\psi](\xi) \qquad (2.19)\\
&= \int_{C_0} F^m(\lambda^{-1/2}x_\sigma + \xi)\,\psi(\lambda^{-1/2}x(t) + \xi)\,dm_w(x).
\end{aligned}$$

By the similar method as in the proof of **Lemma 3** in [1, p527], the limit $\lim_{\|\sigma\|\to 0} (I^\sigma_\lambda(F^m)\psi, \varphi)$ exists and equals $(I_\lambda(F^m)\psi, \varphi)$. And by **Theorem 2.3**

$$\left|(I^\sigma_\lambda(F^m)\psi, \varphi)\right| \qquad \text{is finte.} \tag{2.20}$$

Hence by the Vitali's Theorem [5, p104], $w-\lim I_\lambda^\sigma(F^m) = I_\lambda^{seq}(F^m)$ exists and $(I_\lambda^{seq}(F^m)\psi,\varphi)$ is analytic on $\mathbf{C}_{+,2\pi t,M}$. By the uniqueness theorem in [5], $(I_\lambda^{seq}(F^m)\psi,\varphi) = (I_\lambda^{an}(F^m)\psi,\varphi)$ for λ in $\mathbf{C}_{+,2\pi t,M}$. Since M was arbitrary, $I_\lambda^{seq}(F^m)$ exists and $I_\lambda^{seq}(F^m) = I_\lambda^{an}(F^m)$ for λ in $\mathbf{C}_{+,2\pi t,+\infty}$. The proof of the theorem is completed.

From **Theorem 2.4**, we can easily prove the main theorem in this section.

THEOREM 2.5. *Under the assumptions and notations in the above, the operator-valued Feynman integral $I_\lambda^f(F)$ exists for λ in $\mathbf{C}_{+,2\pi t,+\infty}^{\sim}$ and*

$$\begin{aligned} &I_\lambda^f(F) \qquad\qquad (2.21)\\ &=\sum_{m=0}^{\infty}\sum_{q_0+\cdots+q_h=m}\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\sum_{j_1+\cdots+j_{h+1}=q_0}\\ &\times(BS)-\int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}} L_0\circ L_1\circ\cdots\circ L_h\, d\prod_{i=1}^{q_0}\mu(s_i). \end{aligned}$$

3. The integral equation for the operator-valued feynman integral

In this section, we prove that the operator-valued Feynman integral in the above **Theorem 2.5** is a solution of a related integral equation.

Let t be a given positive real number, $1 < p < 2$ be given, let η be in $M^\sim[0,t]$ and let θ be in $L_{\alpha r;\eta}^{\sim}$. We assume that μ is a continuous part of η and $\nu = \sum_{p=1}^h w_p\delta_{\tau_p}$ is a discrete part of η with $0 < \tau_1 < \tau_2 < \cdots < \tau_h < t$. Denote $\theta_t(s,\cdot) \equiv \theta(t-s,\cdot)$. For u in $(\tau_h, t]$, let

$$G_u(y) = \exp\left(\int_{[0,u]}\theta(u-s, y(u-s))\,d\eta(s)\right) \quad \text{for } y \text{ in } \mathcal{S}, \qquad (3.1)$$

let

$$G_{\tau_h}(y) = \exp\left(\int_{[0,\tau_h)}\theta(\tau_h-s, y(\tau_h-s))\,d\eta(s)\right) \quad \text{for } y \text{ in } \mathcal{S}, \qquad (3.2)$$

and for nonnegative integers $q_0, q_1, \cdots, q_h, j_1, \cdots, j_{h+1}$ and for $(s_1, s_2, \cdots, s_{q_0}) \in \Delta_{q_0; j_1, \cdots, j_{h+1}}$, let

$$\mathcal{L}(q_0, q_1, \cdots, q_h; j_1, \cdots, j_{h+1}; h) \tag{3.3}$$
$$= C_{\lambda/(u-s_{q_0})} \circ \theta_u(s_{q_0}) \circ \cdots \circ C_{\lambda/(s_{j_1+\cdots+j_{h+1}} - \tau_h)} \circ \theta_u(\tau_h)^{q_h} \circ \cdots \circ \theta_u(\tau_1)^{q_1} \circ C_{\lambda/(\tau_1 - s_{j_1})} \circ \cdots \circ \theta_u(s_1) \circ C_{\lambda/s_1}.$$

Then, from **Theorem 2.5**, the operator- valued Feynman integral $I_\lambda^f(G_u)$ exists for λ in $\mathbf{C}^{\sim}_{+,2\pi t,+\infty}$ and

$$I_\lambda^f(G_u) \tag{3.4}$$
$$= \sum_{m=0}^{\infty} \sum_{q_0+\cdots+q_h=m} \frac{w_1^{q_1} \cdots w_h^{q_h}}{q_1! \cdots q_h!} \sum_{j_1+\cdots+j_{h+1}=q_0}$$
$$(BS) - \int_{\Delta_{q_0; j_1, \cdots, j_{h+1}}} \mathcal{L}(q_0, q_1, \cdots, q_h; j_1, \cdots, j_{h+1}; u)\, d \prod_{i=1}^{q_0} \mu(s_i).$$

THEOREM 3.1. *Under the assumptions and notions in the above, we have*

$$I_\lambda^f(G_u) \tag{3.5}$$
$$= C_{\lambda/(u-\tau_h)} \exp[\omega_h \theta_u(\tau_h)] I_\lambda^f(G_{\tau_h})$$
$$+ (BS) - \int_{\tau_h}^{u} C_{\lambda/(u-s)} \circ \theta_u(s) \circ I_\lambda^f(G_s)\, d\mu(s).$$

Proof. From equality (3.4),

$$I_\lambda^f(G_u) \tag{3.6}$$
$$\overset{(1)}{=} C_{\lambda/u} + \sum_{m=0}^{\infty} \sum_{q_0=1}^{m} \sum_{q_0+\cdots+q_h=m-q_0} \frac{w_1^{q_1} \cdots w_h^{q_h}}{q_1! \cdots q_h!} \sum_{j_{h+1}=1}^{q_0} \sum_{j_1+\cdots+j_h=q_0-j_{h+1}}$$
$$\times (BS) - \int_{\Delta_{q_0; j_1, \cdots, j_{h+1}}} \mathcal{L}(u)\, d \prod_{i=1}^{q_0} \mu(s_i)$$
$$+ \sum_{m=0}^{\infty} \sum_{q_0=1}^{m} \sum_{q_1+\cdots+q_h=m-q_0} \frac{w_1^{q_1} \cdots w_h^{q_h}}{q_1! \cdots q_h!} \sum_{j_1+\cdots+j_h=q_0}$$
$$\times (BS) - \int_{\Delta_{q_0; j_1, \cdots, j_h, 0}} \mathcal{L}(u)\, d \prod_{i=1}^{q_0} \mu(s_i)$$

$$\overset{(2)}{=}(BS)-\int_{\tau_h}^{u} C_{\lambda/(u-\tau_h)}\circ\theta_u(s)\circ\Bigg(\sum_{m'=0}^{\infty}\sum_{q_0'=1}^{m'}$$

$$\times\sum_{q_1+\cdots+q_h=m'-q_0'}\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\sum_{j_{h+1}=0}^{q_0'}\sum_{j_1+\cdots+j_h=q_0-j_{h+1}}$$

$$(BS)-\int_{\Delta_{q_0;j_1,\cdots,j_{h+1}}}\mathcal{L}(s)\,d\prod_{i=1}^{q_0'}\mu(s_i)\Bigg)\,d\mu(s)$$

$$+C_{\lambda/(u-\tau_h)}\circ\Bigg(\sum_{m=0}^{\infty}\sum_{q_0=1}^{m}\sum_{q_1+\cdots+q_h=m-q_0}\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\theta_u(\tau_h)^{q_h}$$

$$\sum_{j_1+\cdots+j_h=q_0}(BS)-\int_{\Delta_{q_0;j_1,\cdots,j_h,0}}\mathcal{L}(\tau_h^-)\,d\prod_{i=1}^{q_0}\mu(s_i)\Bigg)$$

$$\overset{(3)}{=}C_{\lambda/(u-\tau_h)}\circ\exp\big(w_h\theta(\tau_h)\big)\circ I_\lambda^f(G_{\tau_h})$$

$$+(BS)-\int_{\tau_h}^{u} C_{\lambda/(s-\tau_h)}\circ\theta(s)\circ I_\lambda^f(G_s)\,d\mu(s).$$

Step (1) and (3) are clear. Putting $m'=m-1$ and $q_0'=q_0-1$, Step (2) follows from the elementary calculus. Therefore, the proof is complete.

REMARK. In the above **Theorem**, if we let η be Lebesgue measure, the equation (3.5) and the equation (5.9) in **Theorem 5.1** of [7, p126] are essentially the same.

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