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LINEAR $p(X) = X$ PRESERVERS OVER GENERAL BOOLEAN SEMIRINGS

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1. Introduction and preliminaries

During the past century, one of the most active and continuing subjects in matrix theory has been the study of those linear operators on matrices that leave certain properties or subsets invariant. Such questions are usually called "Linear Preserver Problems".

Boolean matrices have quite different properties from matrices over a field, due to the fact that addition in a Boolean algebra does not make it a group. Boolean matrices may arise from graphs or from nonnegative real matrices by replacing all positive entries by 1, but their most frequent occurrence is in the representation of binary relations.

Our results in [2] and [3] are stated mostly for Boolean binary (i.e., zero-one) matrices. As K. H. Kim points out in his extensive survey of the Boolean matrix theory [11], there is an isomorphism between the matrices over the Boolean algebra of subsets of a k-element set and the k-tuples of binary Boolean matrices. This isomorphism allows many questions concerning matrices over an arbitrary finite Boolean algebra to be answered using the binary Boolean case. However there are some features of general (i.e., nonbinary) Boolean matrices that have not been mentioned and they differ somewhat from the binary case: for example, it may not be zero divisor free.

In many instances, the extension of results to the general case is not immediately obvious and an explicit version of the above mentioned isomorphism was not well known. In 1992, S. Kirkland and N. J. Pullman [12] gave, in detail, a way to derive results in the general Boolean algebra case via the isomorphism from the binary Boolean algebra case, by means of a canonical form derived from the isomorphism.

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In this paper, we will consider some characterizations of the linear operators that preserve some matrix polynomial equations over finite Boolean algebras to illustrate the differences and similarities of the general *vs* the binary case.

Let $\mathcal{M}_n(\mathbb{B}_k)$ denote the vector space of all $n \times n$ matrices over $\mathbb{B}_k$ where $\mathbb{B}_k$ is the Boolean algebra of subsets of a k-element set $\mathcal{E}_k$ and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the singleton subsets of $\mathcal{E}_k$. As Kirkland and Pullman defined in [12], we let $A \in \mathcal{M}_n(\mathbb{B}_k)$ and write $+$ for union and denote intersection by *juxtaposition*. Under those two operations, $\mathbb{B}_k$ is a commutative, antinegative semiring; all of its entries, except two ($0 = \phi$ and $1 = \mathcal{E}_k$) are zero-divisors.

The i^{th} constituent of $A = [a_{ij}]$, A_i, is the $n \times n$ binary matrix whose $(s,t)^{th}$ entry is 1 if and only if $a_{st} \supseteq \sigma_i$. Evidently the constituent matrices are binary solutions to the equation,

$$A = \sum_{i=1}^{k} \sigma_i X_i \tag{1.1}$$

in the indeterminate $n \times n$ matrices $X_1, X_2, \cdots, X_k$. If $A = \sum_i \sigma_i C_i$ and the C_i are all binary matrices, then $C_i = A_i$ for all $1 \le i \le k$, because (i) the constituent matrices satisfy equation (1.1) and (ii) $\sigma_s \sigma_t = \sigma_s$ or 0 according as $s = t$ or not. Thus the constituents of A are the unique binary solutions to equation. We will refer to $\sum_i \sigma_i A_i$ as the canonical form of A.

For each linear operator $\mathbf{T}$ on $\mathcal{M}_n(\mathbb{B}_k)$ and for each $1 \le i \le k$, define $\mathbf{T}_i : \mathcal{M}_n(\mathbb{B}_1) \to \mathcal{M}_n(\mathbb{B}_1)$ by $\mathbf{T}_i(\bar{X}) = \sigma_i(\mathbf{T}(X))$ where $\bar{X}$ and X are (0, 1) matrices with the same pattern [2], $\bar{X} \in \mathcal{M}_n(\mathbb{B}_1)$ and $X \in \mathcal{M}_n(\mathbb{B}_k)$. $\mathbf{T}_i$ is called the i^{th} constituent of $\mathbf{T}$. Notice that $\mathbf{T}(X) = \mathbf{T}\left(\sum_{i=1}^{k} \sigma_i X_i\right) = \sum_i \sigma_i \mathbf{T}(X_i) = \sum_i \sigma_i \mathbf{T}_i(X_i)$ for any matrix X in $\mathcal{M}_n(\mathbb{B}_k)$.

It follows from the uniqueness of the constituents as binary solutions to (1.1) and the fact that the singletons are mutually orthogonal idempotents that, for each $n \times n$ matrix A, all $n \times r$ matrices B and C, and all $\alpha \in \mathbb{B}_k$,

$$\begin{cases} (a) \quad (AB)_i = A_i B_i, \\ (b) \quad (B + C)_i = B_i + C_i, \text{ and} \\ (c) \quad (\alpha A)_i = \alpha_i A_i, \text{ for all } 1 \le i \le k. \end{cases} \tag{1.2}$$

If x and y are row vectors whose entries x_j, y_j are $n \times n$ binary matrices, let xy be equal to $[x_1y_1, x_2y_2, \cdots, x_ky_k]$, the Hadamard product of the vectors. We define, for each $n \times n$ matrix A over $\mathbb{B}_k$,

$$[A] = \left[A_1, A_2, \cdots, A_k\right],$$

where A_i is the i^{th} constituent of A.

The following theorem is due to S. Kirkland and N. J. Pullman, and is easy to verify.

THEOREM 1.1. *[12] The mapping that sends each matrix A over $\mathbb{B}_k$ to $[A]$, its vector of constituents, is an isomorphism*

For notational purposes we will associate A with its canonical form $\sum_i \sigma_i A_i$ instead of its vector of constituents $[A]$.

The invertible $n \times n$ binary Boolean matrices are all permutation matrices. The invertible matrices over $\mathbb{B}_k$ are orthogonal matrices. This was originally proved by J. H. M. Wedderburn [14]. He also showed that a Boolean matrix is invertible if and only if all its constituents are permutation matrices. Let α^c denote the complement of α for each α in $\mathbb{B}_k$. For $1 \leq i \leq k$, we define the i^{th} rotation operator, $\Theta^{(i)}$, by $\Theta^{(i)}(X) = \sigma_i X^T + \sigma_i^c X$. We see that $\Theta^{(i)}$ has the effect of transposing X_i while leaving the remaining constituents unchanged. Each rotation operator is linear on $\mathcal{M}_n(\mathbb{B}_k)$ and their product is the transposition operator, $\Theta : X \to X^T$.

We now begin our investigation of $p(X) = X$ matrices and their preservers over $\mathbb{B}_k$.

There is an extensive literature concerning characterizations of linear operators that preserves invariant of matrices over rings and fields. A number of analogous results have been obtained for matrices over antinegative semirings by L. B. Beasley, D. A. Gregory and N. J. Pullman in [4, 5, 6, 7, 8, 9, 10].

But many of the proofs of theorems in the above mentioned references use the hypothesis that the antinegative semiring has no zero divisors. So those theorems are not directly applicable to the Boolean algebras of more than two members. However, using the canonical form, the results obtained in the binary case can be exploited to get results valid for arbitrary finite Boolean algebras.

In [12], S. Kirkland and N. J. Pullman showed that $\mathbf{T}$ preserves Boolean rank if and only if it is in the group generated by the rotation and congruence operators. They also gave the following theorem:

THEOREM 1.2. *[12, Theorem 3.2] If $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$, $n \geq 2$, then the following are equivalent.*

(a) *$\mathbf{T}$ strongly preserves idempotence.*
(b) *$\mathbf{T}(A)$ commutes with $\mathbf{T}(B)$ if and only if A commutes with B.*
(c) *$\mathbf{T}$ is in the group generated by the rotation and similarity operators.*

The following theorem, which is a consequence of an important part of the Perron-Frobenius Theorem, will be needed in section 2.

THEOREM 1.3. *If $\mathbb{S}$ is an antinegative semiring and $A \in \mathcal{M}_n(\mathbb{S})$ is an irreducible matrix with index of imprimitivity $h \geq 2$, then there exist a permutation matrix P such that*

$$PA^hP^T = A_1 \oplus A_2 \oplus \cdots \oplus A_h, \text{ a direct sum of } A_i\text{'s},$$

where the A_i are primitive matrices.

For a proof, see [13, Theorem 5.9.4].

In this paper, $(0, 1)$ matrices are in either $\mathbb{B}_k$ or $\mathbb{B}_1$ as needed without special reference. We let $p(X) = X^{r_1} + X^{r_2} + \cdots + X^{r_s}$ where $r_1 > r_2 > \cdots > r_s \geq 2$, $d = gcd(r_1, r_2, \cdots, r_s)$, and assume $d \geq 2$. Also let $\mathbb{A}$ be an antinegative semiring with no zero divisors, $n \geq r \geq 2$ and $\mathbf{T}$ be a linear operator on $\mathcal{M}_n = \mathcal{M}_n(\mathbb{A})$ which strongly preserves the set $\mathcal{X} = \{X \in \mathcal{M}_n(\mathbb{A}) | p(X) = X\}$.

2. The Boolean $(0,1)$ case

Recall that a matrix X is said to be r-potent if $X^r = X$. Throughout this section, $\mathbb{A} = \mathbb{B}_1$, the Boolean algebra of two elements, and all matrices are in $\mathcal{M} = \mathcal{M}_n(\mathbb{B}_1)$. In this section, we will characterize all linear operators that strongly preserve the polynomial equation $p(X) = X$ and we will assume $r > s \geq 2$.

LEMMA 2.1. *A matrix X satisfies $X^r = X$ and $X^s = X$ if and only if $X^\alpha = X$ where $\alpha = gcd(r-1, s-1) + 1$.*

Proof. Suppose $X^\alpha = X$ and $\alpha - 1 = gcd(r-1, s-1)$. Then $r - 1 = k(\alpha - 1)$ for some positive integer $k \geq 2$ since $r > s \geq 2$. Therefore $X^{r-1} = X^{k(\alpha-1)}$. This implies $X^r = X^{k(\alpha-1)}X = (X^{\alpha-1})^k X = (X^{\alpha-1})^{k-1}X^{\alpha-1}X = (X^{\alpha-1})^{k-1}X = (X^{\alpha-1})^{k-1}X$. Repeat this process until $k-1 = 1$, then we will have $X^r = X^{k(\alpha-1)}X = X^{\alpha-1}X = X$. Similarly we will have $X^s = X$.

Suppose $X^r = X^s = X$, $r > s$, and $\alpha = gcd(r-1, s-1) + 1$. Then there are integers a and b such that $a(r-1) + b(s-1) = \alpha - 1$. If $a = 0$ or $b = 0$, either $r - 1$ or $s - 1$ is $gcd(r-1, s-1)$ and the Lemma follows. Therefore we may assume $a \neq 0$ and $b \neq 0$. Since $r > s \geq 2$, only one of a and b is positive. Suppose $a > 0$, $b < 0$. First we note that $a - b > 0$, $-b > 0$ and hence $a(r-1) = (\alpha - 1) - b(s-1)$. Therefore $X^{ar-a} = X^{(\alpha-1)-b(s-1)}$. This implies $X^a = X^{ar} = X^{ar-a}X^a = X^{(\alpha-1)}X^{-bs+b}X^a$. Multiply X^{-b} on both sides, then $X^{a-b} = X^{\alpha-1}X^{-bs}X^a = X^{\alpha-1}X^{-b}X^a = X^{\alpha-1}X^{a-b}$ since $X^s = X$.

If $a - b \leq r$, then there is a nonnegative integer $t \geq 0$ such that $a - b + t = r$. Therefore $X^r = X^{a-b+t} = X^{a-b}X^t = X^{\alpha-1}X^{a-b}X^t = X^{\alpha-1}X^{a-b+t} = X^{\alpha-1}X^r = X^{\alpha-1}X = X^\alpha$, so $X^\alpha = X^r = X$.

If $a - b > r$, then there is a positive integer t such that $a - b = r + t$. This implies $X^{r+t} = X^{a-b} = X^{\alpha-1}X^{a-b} = X^{\alpha-1}X^{r+t}$. Therefore $X^{t+1} = X^{\alpha-1}X^{t+1}$. Choose q so that $t < r^q$. Then for some $u \geq 1$, $r^q = t + u$, and $X = X^{r^q} = X^{t+1}X^{u-1} = X^{\alpha-1}X^{t+1}X^{u-1} = X^{\alpha-1}X^{r^q} = X^\alpha$. That is $X^\alpha = X$. The argument if $b > 0$ and $a < 0$ is parallel.

LEMMA 2.2. *A matrix X satisfies $X^r + X^s = X$ if and only if X is both r-potent and s-potent.*

Proof. The necessity is trivial. Now we suppose $X^r + X^s = X$. Then $X^r \leq X$ and $X^s \leq X$. Now we only need to show $|X^r| \geq |X|$ and $|X^s| \geq |X|$. By restricting our attention to the irreducible diagonal blocks, we may assume, without loss of generality, that X is an irreducible matrix with index of imprimitivity h. If $X = J$, the conclusion follows immediately since X is idempotent. Suppose $X \neq J$. Then X is not primitive since $X^r + X^s = X$. Therefore the index of imprimitivity is strictly greater than 1. Now by Theorem 1.3, there is a permutation matrix P such that $P\ X^h P^T = X_1 \oplus X_2 \oplus \cdots \oplus X_h$ where X_i's are primitive matrices. Therefore there is an integer N such that $[P\ X^h P^T]^t \geq I$ for all $t \geq N$. Without loss of generality, assume $X^{ht} \geq I$ for all $t \geq N$.

Since $X^r + X^s = X$ and h is the index of imprimitivity of X, there is an integer a such that $r = ah+1$. Now choose m so that $a^m \cdot h^{m-1} \geq N$, and let $q = a^m \cdot h^{m-1}$. Then $X^{r^m} \geq X^{hq+1} X^{hq} X$. Therefore $X^{r^m} \geq X$ since $X^{hq} \geq I$. Since $X^r + X^s = X, (X^r + X^s)^r + (X^r + X^s)^s = X$ and hence $X^{r^2} \leq X$. Repeating, we get $X^{r^c} \leq X$ for all c. In particular $X^{r^m} \leq X$ and $X^{r^{m-1}} \leq X$.

It follows that $X^{r^m} = X$. Now $X = X^{r^m} = (X^r)^{r^{m-1}} \leq X^{r^{m-1}}$ since $X^r \leq X$. Since $X^{r^{m-1}} \leq X$ we get $X = X^{r^{m-1}}$. Therefore $X^r = \left(X^{r^{m-1}}\right)^r = X^{r^m} = X$. Similarly $X^s = X$.

LEMMA 2.3. *The linear operator* $\mathbf{T}$ *strongly preserves* $X^r + X^s = X$ *if and only if* $\mathbf{T}$ *strongly preserves* $X^\alpha = X$ *where* $\alpha = gcd(r-1, s-1)+1$.

Proof. By the Lemma 2.1 and 2.2, the lemma follows immediately.

LEMMA 2.4. *The linear operator* $\mathbf{T}$ *strongly preserves* $p(X) = X$ *if and only if* $\mathbf{T}$ *strongly preserves* $X^\alpha = X$ *where* $\alpha - 1 = gcd(r_1 - 1, r_2 - 1, \cdots, r_t - 1)$.

Proof. Since $gcd[gcd(a-1, b-1), c-1] = gcd(a-1, b-1, c-1)$, it follows directly from Lemma 2.3.

THEOREM 2.1. *The semigroup,* φ, *of linear operators strongly preserving* $p(X) = X$ *is generated by transposition and the similarity operators.*

Proof. Since φ is same as the semigroup of linear operators strongly preserving $X^\alpha = X$, where $\alpha = gcd(r_1 - 1, r_2 - 1, \cdots, r_t - 1) + 1$, the conclusion follows immediately from Theorem 3.1 in [2] and Lemma 2.4.

3. Linear $p(X) = X$ preservers over general Boolean semirings

We start with the following lemma in order to generalize Theorem 2.1 to the general finite Boolean algebra case. We also extend Theorem 1.2.

LEMMA 3.1. *Let $Y \in \mathcal{M}_n(\mathbb{B}_k)$. Then Y is r-potent if and only if each constituent $Y_i, 1 \leq i \leq k$, is r-potent.*

Proof. Suppose $Y = \sum_i \sigma_i Y_i$ is r-potent Then by observations (1.2), we have $Y^r = \sum_i \sigma_i Y_i^r$. Thus, $\sum_i \sigma_i Y_i = \sum_i \sigma_i Y_i^r$ so that $Y_i = Y_i^r$ for each i. Similarly if $Y_i = Y_i^r$ for each i, then $Y = Y^r$.

LEMMA 3.2. *Suppose $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$. Then $\mathbf{T}$ strongly preserves r-potence if and only if each constituent operator $\mathbf{T}_i$, $1 \leq i \leq k$, strongly preserves r-potence on $\mathcal{M}_n(\mathbb{B}_1)$.*

Proof. Suppose that $\mathbf{T}$ strongly preserves r-potence on $\mathcal{M}_n(\mathbb{B}_k)$ and Y_i, $1 \leq i \leq k$, is r-potent. Then $Y = \sum_i \sigma_i Y_i$ is r-potent and hence $\mathbf{T}(Y) = \sum_i \sigma_i \mathbf{T}_i(Y)$ is r-potent. That is, each $\mathbf{T}_i(Y)$ is r-potent by Lemma 3.1 and hence $\mathbf{T}_i$ preserves r-potence. If Y_i is not r-potent, let $Y_j = I$ for $j \neq i$. Then Y is not r-potent but $\mathbf{T}_j(Y_j)$ is r-potent for all $j \neq i$. Thus $\mathbf{T}_i(Y_i)$ is not r-potent, for otherwise $\mathbf{T}(Y)$ would be r-potent, an impossibility. Thus $\mathbf{T}_i$ strongly preserves r-potence.

Now, suppose each constituent $\mathbf{T}_i$, $i = 1, \cdots, k$, strongly preserves r-potence. Then Y is r-potent if and only if Y_i is r-potent, $i = 1, \cdots, k$, if and only if $\mathbf{T}_i(Y_i)$ is r-potent, $i = 1, \cdots, k$, if and only if $\mathbf{T}(Y) = \sum_i \sigma_i \mathbf{T}_i(Y_i)$ is r-potent. That is, $\mathbf{T}$ strongly preserves r-potence.

LEMMA 3.3. *Suppose $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$. Then $\mathbf{T}$ strongly preserves the polynomial equation $p(X) = X$ over $\mathbb{B}_k$ if and only if each constituent $\mathbf{T}_i$, $1 \leq i \leq k$, strongly preserves the polynomial equation $p(X) = X$ over $\mathbb{B}_1$.*

Proof. This follows from Lemmas 2.4 and 3.2.

THEOREM 3.1. *Suppose $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$. Then $\mathbf{T}$ strongly preserves the polynomial equation $p(X) = X$ if and only if $\mathbf{T}$ is in the group generated by the rotation and similarity operators.*

Proof. Since all rotation and similarity operators preserve $p(X) = X$ strongly, we only need to show any linear operator on matrices over

$\mathbb{B}_k$ that strongly preserves $p(X) = X$, is generated by those two sets of operators. But this follows from Lemma 3.3, Theorem 2.1, and the remarks on rotation operators.

Therefore we have the following generalization of Theorem 1.2.

COROLLARY 3.1. *If* $\mathbf{T}$ *is a linear operator on* $\mathcal{M}_n(\mathbb{B}_k)$, $n \geq 2$, *then the followings are equivalent.*

(a) $\mathbf{T}$ *strongly preserves the polynomial equation* $p(X) = X$.
(b) $\mathbf{T}(A)$ *commutes with* $\mathbf{T}(B)$ *if and only if* A *commutes with* B.
(c) $\mathbf{T}$ *is in the group generated by the rotation and similarity operators.*

4. Linear $p(X) = X$ preservers over antinegative semirings

Rings and fields are examples of semirings, but there are also some semirings which are not rings. One good example is an antinegative semiring. No ring with unity can be antinegative, but many interesting structures, such as the nonnegative integers, the nonnegative reals, and the Boolean algebras are antinegative semirings which occur in combinatorics.

In [2], [3] and the previous section, we have given characterizations of linear operators on matrices that preserve several polynomial equations over the binary Boolean semiring and general Boolean semirings. Our next step is to extend those results to linear operators on matrices over any antinegative semiring with no zero divisors such as nonnegative integers, and nonnegative reals. The mapping accomplished by associating each matrix, A, in $\mathcal{M}_n(\mathbb{A})$ with its pattern, $\overline{A}$, in $\mathcal{M}_n(\mathbb{B}_1)$ is a semiring homomorphism when $\mathbb{A}$ is antinegative and zero-divisor-free.

If $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{A})$, let $\overline{\mathbf{T}}$, its pattern, be the operator on $\mathcal{M}_n(\mathbb{B}_1)$ defined by $\overline{\mathbf{T}}(\overline{E_{ij}}) = \overline{\mathbf{T}(E_{ij})}$ for all (i,j). Then $\overline{\mathbf{T}(A)} \leq \overline{\mathbf{T}}(\overline{A})$ for all A in $\mathcal{M}_n(\mathbb{A})$. Equality holds if $\mathbb{A}$ is an antinegative semiring having no zero divisors.

LEMMA 4.1. *[10, Lemma 1.1] The mapping* $\mathbf{T} \to \overline{\mathbf{T}}$ *is a homomorphism of the semigroup of linear operators on* $\mathcal{M}_n(\mathbb{A})$ *onto the semigroup of linear operators on* $\mathcal{M}_n(\mathbb{B}_1)$.

Let $A \in \mathcal{M}_n(\mathbb{A})$. Recall that the scaling operator, $\mathbf{L}_A$, induced by A, was defined by $\mathbf{L}_A : X \to A \circ X$.

Throughout this section, let $\varphi = \varphi_n(\mathbb{A})$ denote the semigroup of all linear operators on $\mathcal{M}_n(\mathbb{A})$ strongly preserving r-potence. Let $\varphi' = \varphi'(\mathbb{A})$ denote the semigroup of all linear operators on $\mathcal{M}_n(\mathbb{A})$ strongly preserving polynomial equation $p(X) = X$. Let $d = gcd(r_1 - 1, r_2 - 1, \cdots, r_t - 1) + 1$.

LEMMA 4.2. *The semigroups φ and φ' are generated by the scaling operators in φ, transposition and the similarity operators.*

Proof. Suppose $\mathbf{T} \in \varphi$, (resp. φ'). Then $\overline{\mathbf{T}} \in \varphi_n(\mathbb{B}_1)$ (resp. $\varphi'_n(\mathbb{B}1)$) since $\overline{\mathbf{T}(X)} = \overline{\mathbf{T}}(\overline{X})$ whenever $\mathbb{A}$ is an antinegative semiring having no zero divisors. Therefore $\overline{\mathbf{T}}$ is in the semigroup of operators generated by the similarity operators and transposition by Theorem 2.2 (resp. Theorem 3.2) in [2]. Thus $\mathbf{T}(X) = \mathbf{M} \circ \overline{\mathbf{T}(A)}$ for some $M \in \mathcal{M}$ and the lemma follows.

For some semirings, we can characterize the scaling operators that strongly preserve r-potence or $p(X) = X$.

LEMMA 4.3. *If $n \geq 3$ and every element of $\mathbb{A}$ is idempotent, then the identity operator is the only scaling operator that strongly preserves r-potence.*

Proof. Clearly, the identity operator is $\mathbf{L}_J$. Suppose $\mathbf{L} = \mathbf{L}_A$ strongly preserves r-potence for some A. Let i, j and k be distinct positive integers, $i, j, k \leq n$. Put $X_{ijk} = a_{ij}E_{ij} + E_{ik} + E_{jj} + E_{jk}, J_{ijk} = E_{ij} + E_{ik} + E_{jj} + E_{jk}, X_{jk} = a_{jj}E_{jj} + E_{jk}$, and $J_{jk} = E_{jj} + E_{jk}$. It is easily seen that J_{ijk} and J_{jk} are r-potnet. Since $\mathbf{L}(X_{ijk}) = \mathbf{L}(J_{ijk})$ and $\mathbf{L}(X_{jk}) = \mathbf{L}(J_{jk})$, we have that X_{ijk} and X_{jk} are r-potent. Then the (i, k) entry of $(X_{ijk})^r$ is a_{ij} while the (i, k) entry of X_{ijk} is 1. Thus, $a_{ij} = 1$. Also, the (j, k) entry of $(X_{jk})^r$ is a_{jj} while the (j, k) entry of X_{jk} is 1. Thus, $a_{jj} = 1$. Since i, j were arbitrary chosen, we have $A = J$.

Note: In Lemma 4.3, $\mathbb{A}$ need not be antinegative.

THEOREM 4.1. *If $n \geq 3$ and every member of $\mathbb{A}$ is idempotent then φ is generated by transposition and the similarity operators; φ is therefore a group.*

Proof. This is immediate from Lemmas 4.1 and 4.2.

The permutation matrices are the only invertible matrices over those antinegative semirings that have only one unit 1, such as the nonnegative integers, any chain semiring, such as the fuzzy scalars, or the two element Boolean algebra. For any antinegative semiring $\mathbb{A}, Q$ is invertible in $\mathcal{M}_n(\mathbb{A})$ if and only if $Q = PD$ for some permutation matrix P and some diagonal matrix D whose diagonal entries are all units in the antinegative semiring $\mathbb{A}$.

COROLLARY 4.1. *If $n \geq 3$, the semigroup of linear operators on the $n \times n$ matrices over any Boolean algebra or chain semiring, that strongly preserves a polynomial equation $p(X) = X$, is generated by transposition and the operators $X \rightarrow PXP^T, P$ a permutation matrix.*

LEMMA 4.4. *If $\mathbf{L}_A$ preserves r-potence on $\mathcal{M}_n(\mathbb{A})$, then each diagonal entry in A is r-potent.*

Proof. Since $I^r = I$ we must have that $\mathbf{L}_A(I)$ is r-potent. Thus $(A \circ I)^r = [\mathbf{L}_A(I)]^r = \mathbf{L}_A(I) = A \circ I$, and thus $a_{ii}^r = a_{ii}$ for all $i,\ 1 \leq i \leq n$.

The antinegative semirings of combinatorial interest are mostly those with Boolean arithmetic, i.e. $(\mathbb{A}, \cup, \cap)$ where $\cup = +$ and $\cap = \cdot$, and those with real arithmetic, i.e. subsemirings of $\mathbb{R}^+$. We note that all these semirings are examples of semirings where 1 is the only r-th root of unity in it. We investigate $\varphi(\mathbb{A})$ when $\mathbb{A}$ has one of these types of arithmetic.

LEMMA 4.5. *Suppose $\mathbb{A}$ is an antinegative, commutative semiring with only one $(r-1)^{th}$ root of unity, 1, having the multiplicative cancellation property.*

(i) *If $\mathbf{L}_A$ strongly preserves r-potence, then*
  (a) *each diagonal entry in A is 1, and*
  (b) *when $n \geq 3, \mathbf{L}_A$ is a unit scaling operator*

(ii) *If $a_{ij} = a_i a_j^{-1}$ for some invertible a_i, a_j in $\mathbb{A}$, for all i, j, then $\mathbf{L}_A$ strongly preserves r-potence.*

Proof. Since each diagonal entry in A is an r-potent (Lemma 4.4) and none are 0 by Theorem 2.1 in [2], it follows by the cancellation property, that $a_{ii}^{r-1} = 1$. This implies $a_{ii} = 1$ for all i since $\mathbb{A}$ has only one $(r-1)^{th}$ root of unity, 1. This establishes (i) (a).

Next we fix i and choose $j \neq i$. Let $R = \sum_{k\neq i}(E_{ik} + E_{jk})$, then by direct computation we have $(A \circ R)^2 = (A \circ R)^3$ and hence $(A \circ R)^m = (A \circ R)^2$ for all $m > 1$. In particular, $(A \circ R)^r = (A \circ R)^2$. But R is r-potent so its image, $A \circ R = \mathbf{L}_A(R)$, is r-potent too. Consequently $A \circ R = (A \circ R)^r = (A \circ R)^2$, i.e. $A \circ R$ is idempotent.

Therefore

$$\sum_{k\neq i}(a_{ik}E_{ik} + a_{jk}E_{jk}) = \sum_{k\neq i}(a_{ik}a_{jk}E_{ik} + a_{jk}E_{jk}).$$

Therefore

$$a_{ik} = a_{ij}a_{jk}, \qquad \text{for all} \quad k \neq i, \tag{4.1}$$

and by interchanging the roles of i and j in (3 1), we obtain

$$a_{jk} = a_{ji}a_{ik}, \qquad \text{for all} \quad k \neq j, \tag{4.2}$$

Since $n \geq 3$ we can choose $g \neq i, j$ obtaining

$$a_{ig} = a_{ij}a_{ji}a_{ig} \tag{4.3}$$

from (4.1) and (4.2). Suppose $a_{ig} = 0$, then $X + E_{ii} + E_{ij} + E_{ig} + E_{jg} + E_{gg}$ is r-potent while $Y = E_{ii} + E_{ij} + E_{ja} + E_{gg}$ is not. But $A \circ X = A \circ Y$, contradicting that $\mathbf{L}_A$ strongly preserves r-potence. Thus $a_{ij} \neq 0$ for all i, j. Hence from (4.3), $1 = a_{ij}a_{ji}$ so that a_{ij} is a unit for all i, j. From (4.1) we have $a_{ig} = a_{ij}a_{jg}$ so that $a_{ij} = a_{ig}a_{jg}^{-1}$. Let $a_1 = a_{12}$ and $a_k = a_{k1}$ for all $k \geq 2$. This completes the proof of part (i).

The verification of part (ii) is a straight-forward computation.

Let $\mathbb{P}^+$ be the nonnegative members of a nontrivial subring $\mathbb{P}$ of the reals. That is, if $\mathbb{P} = \mathbb{R}$ (reals) then $\mathbb{P}^+ = \mathbb{R}^+$, if $\mathbb{P} = \mathbb{Z}$ (integers) then $\mathbb{P}^+ = \mathbb{Z}^+$.

Note : If $\mathbb{A} = \mathbb{Z}^+$, then Lemma 4.5 (i) implies that all $a_{ij} = 1$.

THEOREM 4.2. *The semigroup $\varphi = \varphi_n(\mathbb{P}^+)$ os generated by transposition and the similarity operators, unless $n = 2$ and $\mathcal{M}_2(\mathbb{P}^+)$'s r-potent are triangular and hence are on a single line. In that case, an additional family of generators is required, namely, the set of scaling operators $X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X$ with $xy > 0$.*

Proof. Because a scaling operator induced by a matrix A satisfying the codition, Lemma 4.5 (i), is the similarity operator $X \to DXD^{-1}$ where $D = diag(a_1, a_2, \cdots, a_n)$, the theorem is immediate from Theorem 2.2 in [2] and Lemma 2.2 in [2] unless $n = 2$ and $\mathcal{M}_2(\mathbb{P}^+)$'s r-potents are triangular. In that case, suppose $\mathbf{T}$ is in φ. Lemma 3.3 implies that we may assume $\mathbf{T}$ is a scaling operator, say $\mathbf{T} = \mathbf{L}_A$. According to Lemma 4.5, $A = \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix}$ for some x, y in $\mathbb{P}^+$. Then $xy > 0$, otherwise $A \circ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is r-potent while $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is not, a contradiction. Conversely, by observation the scaling operators $X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X$ strongly preserves r-potent matrices.

COROLLARY 4.2. *The semigroup $\varphi = \varphi_n(\mathbb{R}^+)$ is generated by transposition, permutation similarity and $X \to D\ X\ D^{-1}$ where D is a diagonal matrix and all $d_{ii} > 0$.*

COROLLARY 4.3. *The semigroup $\varphi = \varphi_n(\mathbb{Z}^+)$ is generated by transposition and permutation similarity, unless $n = 2$ and r is odd. If $n = 2$ and r is odd, an additional family of generators is needed, namely all the scaling operators $X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X$ with $xy \geq 1$.*

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