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GENERIC SUBMANIFOLDS OF A QUATERNIONIC KAEHLERIAN MANIFOLD WITH NONVANISHING PARALLEL MEAN CURVATURE VECTOR

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0. Introduction

A submanifold M of a quaternionic Kaehlerian manifold $\tilde{M}^m$ of real dimension $4m$ is called a *generic submanifold* if the normal space $N(M)$ of M is always mapped into the tangent space $T(M)$ under the action of the quaternionic Kaehlerian structure tensors of the ambient manifold at the same time.

The purpose of the present paper is to study generic submanifold of quaternionic Kaehlerian manifold of constant Q-sectional curvature with nonvanishing parallel mean curvature vector.

In section 1, we state general formulas on generic submanifolds of a quaternionic Kaehlerian manifold of constant Q-sectional curvature. Section 2 is devoted to the study generic submanifolds with nonvanishing parallel mean curvature vector and compute the restricted Laplacian for the second fundamental form in the direction of the mean curvature vector. As applications of those results, in section 3, we prove our main theorems. In this paper, the dimension of a manifold will always indicate its real dimension.

1. Preliminaries

Let $i : M \longrightarrow \tilde{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+p)$-dimensional Riemannian manifold $\tilde{M}$ with Riemannian metric g. We denote by the same g the Riemannian metric induced on M from that of $\tilde{M}$.

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Let ∇ and $\tilde{\nabla}$ be the Riemannian connections on M and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas are respectively given by

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + B(X,Y), \\ \tilde{\nabla}_X V &= -A_V X + D_X V \end{aligned} \tag{1.1}$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A and B appearing here are both called the *second fundamental forms* of M and are related by

$$g(B(X,Y),V) = g(A_V X, Y). \tag{1.2}$$

The second fundamental form A_V in the direction of the normal vector V can be considered as a symmetric (n,n)-matrix.

The covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y. \tag{1.3}$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of V*. If the second fundamental form is parallel in any direction, it is said to be parallel. This is equivalent to $(\nabla_X B)(Y,Z) = 0$ for any vector fields X, Y and Z tangent to M, where we have put

$$(\nabla_X B)(Y,Z) = D_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

The mean curvature vector ν of M is defined to be $\nu = \frac{1}{n} TrB$, where TrB denotes the trace of B. If $\nu = 0$, then M is said to be *minimal*. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M. A parallel normal vector field $V(\neq 0)$ is called an *isoperimetric* section if TrA_V is constant.

Let $\tilde{M}$ be an $(n+p)$-dimensional quaternionic Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}\}$ and its quaternionic Kaehlerian structure be denoted by $(g, \tilde{V})$(see,[1]).

Then there exists a canonical local basis $\{F, G, H\}$ of the 3-dimensional vector bundle $\tilde{V}$ consisting of tensors of type (1,1) over M such that

$$\begin{aligned} F^2 = G^2 = H^2 &= -I, \\ GH = -HG &= F, \\ HF = -FH &= G, \\ FG = -GF &= H \end{aligned} \tag{1.4}$$

in each local coordinate neighborhood $\tilde{U}$, where I denotes the identity tensor field. Moreover, the local tensor fields F, G and H are almost Hermitian with respect to g and equations

$$\begin{aligned} \tilde{\nabla}_{\tilde{X}} F &= r(\tilde{X})G - q(\tilde{X})H, \\ \tilde{\nabla}_{\tilde{X}} G &= -r(\tilde{X})F + p(\tilde{X})H, \\ \tilde{\nabla}_{\tilde{X}} H &= q(\tilde{X})F - p(\tilde{X})G \end{aligned} \tag{1.5}$$

are satisfied for any vector field $\tilde{X}$ on $\tilde{M}$, where p, q, r are local 1-forms defined on $\tilde{U}$.

When $\tilde{M}^m$ is a $4m$-dimensional quaternionic Kaehlerian manifold of constant Q-sectional curvature c, $\tilde{M}^m$ is said to be a quaternionic space form, we denote such a space by $\tilde{M}^{4m}(c)$. It is well-known that a quaternionic space form $\tilde{M}^{4m}(c)$ has the following curvature form

$$\begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{c}{4}\Big\{ & g(\tilde{Y}, \tilde{Z})\tilde{X} - g(\tilde{X}, \tilde{Z})\tilde{Y} + \sum_{r=1}^{3}(g(\psi_r\tilde{Y}, \tilde{Z})\psi_r\tilde{X} \\ & - g(\psi_r\tilde{X}, \tilde{Z})\psi_r\tilde{Y} - 2g(\psi_r\tilde{X}, \tilde{Y})\psi_r\tilde{Z})\Big\}, \end{aligned} \tag{1.6}$$

where $\psi_1 = F, \psi_2 = G$, and $\psi_3 = H$.

If the transforms by F, G and H of any vector normal to M are always tangent to M at the same time, then the submanifold M is called a *generic submanifold.* In such a case, $n \geq 3p$, because the ranks of F, G and H are all $n + p$.

For a generic submanifold M of $\tilde{M}^{4m}(c)$, we can put

$$\psi_r X = P_r X + F_r X \ (r = 1, 2, 3) \tag{1.7}$$

for any tangent vector X on M, where P_rX and F_rX are the tangential and normal parts of ψ_rX, respectively.

Similarly, we put

$$\psi_r V = t_r V \quad (r = 1,2,3) \tag{1.8}$$

for any vector field V normal to M, where t_rV is the tangential vector on M.

It is clear from (1.4), (1.7) and (1.8) that

$$\begin{aligned} &g(P_rX,Y) + g(X,P_rY) = 0, \\ &g(F_rX,V) + g(X,t_rV) = 0 \quad (r = 1,2,3) \end{aligned} \tag{1.9}$$

for any vector fields V normal to M. Moreover, we obtain

$$\begin{cases} P_r^2 = -I - t_rF_r, \quad F_rP_r = o, \quad P_rt_r = 0, \\ F_rt_s = \begin{cases} -I, & r = s \\ 0, & r \neq s \ (r,s = 1,2,3) \end{cases} \\ P_2P_3 + t_2F_3 = P_1, \ P_3P_2 + t_3F_2 = -P_1, \\ P_3P_1 + t_3F_1 = P_2, \ P_1P_3 + t_1F_3 = -P_2, \\ P_1P_2 + t_1F_2 = P_3, \ P_2P_1 + t_2F_1 = -P_3, \\ F_2P_3 = F_1, \ F_3P_1 = F_2, \ F_1P_2 = F_3, \\ F_3P_2 = -F_1, \ F_1P_3 = -F_2, \ F_2P_1 = -F_3, \\ P_2t_3 = t_1, \ P_3t_1 = t_2, \ P_1t_2 = t_3, \\ P_3t_2 = -t_1, \ P_1t_3 = -t_2, \ P_2t_1 = -t_3. \end{cases} \tag{1.10}$$

Differentiating covariantly (1.7), (1.8), and using (1.1), (1.5), we can easily see that

$$\begin{aligned} (\nabla_Y P_1)X &= A_{F_1X}Y + r(Y)P_2X - q(Y)P_3X + t_1B(X,Y) \\ (\nabla_Y P_2)X &= A_{F_2X}Y + p(Y)P_3X - r(Y)P_1X + t_2B(X,Y) \\ (\nabla_Y P_3)X &= A_{F_3X}Y + q(Y)P_1X - p(Y)P_2X + t_3B(X,Y), \end{aligned} \tag{1.11}$$

$$\begin{aligned} (D_YF_1)X &= r(Y)F_2X - q(Y)F_3X - B(P_1X,Y) \\ (D_YF_2)X &= p(Y)F_3X - r(Y)F_1X - B(P_2X,Y) \\ (D_YF_3)X &= q(Y)F_1X - p(Y)F_2X - B(P_3X,Y), \end{aligned} \tag{1.12}$$

$$A_W t_r V = A_V t_r W \text{ equivalently } B(Y, t_r V) = -F_r(A_V Y) \tag{1.13}$$

for any vector field X, Y tangent to M and V, W normal to M.

From(1.6), we can easily see that the structure equations of Gauss, Codazzi and Ricci are respectively given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\Big\{ & g(Y,Z)X - g(X,Z)Y + \sum_{r=1}^{3}(g(P_rY,Z)P_rX \\ & - g(P_rX,Z)P_rY - 2g(P_rX,Y)P_rZ)\Big\} \\ & + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \end{aligned} \tag{1.14}$$

$$\begin{aligned} & g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) \\ & = \frac{c}{4}\sum_{r=1}^{3}\{g(P_rY,Z)g(F_rX,V) - g(P_rX,Z)g(F_rY,V) \\ & - 2g(P_rX,Y)g(F_rZ,V)\}, \end{aligned} \tag{1.15}$$

$$\begin{aligned} & g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X,Y) \\ & = \frac{c}{4}\sum_{r=1}^{3}\{g(F_rY,V)g(F_rX,U) - g(F_rX,V)g(F_rY,U)\}, \end{aligned} \tag{1.16}$$

where R denotes the Riemannian curvature tensor of M and $R^{\perp}$ the curvature tensor of the normal bundle of M given by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V$$

for any vector fields X, Y tangent to M and V normal to M.

2. Parallel mean curvature vector field

Let M be an n-dimensional generic submanifold of an $(n+p)$-dimensional quaternionic space form $\tilde{M}^{n+p}(c)$ with nonvanishing parallel mean curvature vector field ν which satisfies

$$B(P_rX, Y) + B(P_rY, X) = 0 \tag{2.1}$$

or equivalently

$$A_V P_r X - P_r A_V X = 0 \tag{2.1'}$$

for any vector fields X and Y tangent to M and V normal to M.

We set $\mu = \frac{\nu}{\|\nu\|}$. Then μ is a nonvanishing parallel unit normal vector, namely, μ is an isoperimetric section in the normal bundle.

We take an orthonormal basis $e_1, \ldots, e_{n+p}$ of $\tilde{M}^{n+p}(c)$ such that $e_1, \ldots, e_n$ form an orthonormal basis of M and $e_{n+1}, \ldots, e_{n+p}$ form an orthonormal basis of $TM^{\perp}$ with $e_{n+1} = \mu, TrA_a = 0$ $(a = n+2, \ldots, n+p)$. Unless otherwise stated, we use the conventions that the ranges of indices are respectively: $i, j, k, \cdots = 1, \ldots, n; a, b, c, \cdots = n+1, \ldots, n+p$.

LEMMA 2.1. *The second fundamental forms of M satisfy*

$$g(A_\mu X, A_V Y) = \sum_a g(A_a X, Y) g(A_\mu t_1 V, t_1 \mu) + \frac{c}{4}\Big\{g(X,Y)g(V,\mu) - \sum_{r=1}^{3} g(X, t_r\mu) g(Y, t_r V)\Big\}, \tag{2.2}$$

where A_a denotes the second fundamental form in the diretion of e_a.

Proof. By the assumption, we have $A_\mu P_r = P_r A_\mu$ (r=1,2,3). Hence, in particular

$$g(A_\mu P_1 X, t_1 V) = 0$$

for any vector field X tangent to M and any vector field V normal to M. We then have

$$g((\nabla_Y A)_\mu P_1 X, t_1 V) + g(A_\mu(\nabla_Y P_1)X, t_1 V) + g(A_\mu P_1 X, (\nabla_Y t_1)V) = 0,$$

from which, using (1.11) and (1.12),

$$g((\nabla_{P_1 Y} A)_\mu P_1 X, t_1 V) + g(A_\mu A_{F_1 X} P_1 Y, t_1 V) + g(t_1 B(X, P_1 Y), A_\mu t_1 V) - g(A_\mu P_1 X, A_V P_1^2 Y) = 0.$$

Using equation of Codazzi, we see

$$\begin{aligned} &-\frac{c}{4}\{2g(X, P_1 Y)g(V,\mu) + g(Y, t_2 V) g(F_3 X, \mu) \\ &- g(X, t_2 V) g(F_3 Y, \mu) + g(Y, t_3 V) g(F_2 X, \mu) - g(X, t_3 V) g(F_2 Y, \mu)\} \\ &= 2g(t_1 B(X, P_1 Y), A_\mu t_1 V) + g(A_\mu Y, A_V P_1 X) - g(A_\mu X, A_V P_1 Y). \end{aligned}$$

Using equation of Ricci, this is

$$-\frac{c}{4}\{g(X,P_1Y)g(V,\mu)+g(Y,t_2V)g(F_3X,\mu)-g(Y,t_3V)g(F_2X,\mu)\}$$
$$=g(t_1B(X,P_1Y),A_\mu t_1V)+g(A_VY,A_\mu P_1X).$$

Since, using (1.13)

$$g(t_1B(P_1X,P_1Y),A_\mu t_1V)=\sum_a g(A_aX,Y)g(A_at_1V,t_1\mu) - g(A_{F_1Y}A_\mu t_1V,t_1F_1X),$$

$$g(A_VY,A_\mu P_1^2X)=-g(A_\mu X,A_VY)+g(A_\mu A_{F_1Y}t_1V,t_1F_1X),$$

from which, we have

$$g(A_\mu X,A_VY)=\frac{c}{4}\{g(P_1X,P_1Y)g(V,\mu)-g(X,t_2\mu)g(Y,t_2V) - g(X,t_3\mu)g(Y,t_3V)\}+g([A_\mu,A_{F_1Y}]t_1V,t_1F_1X) + \sum_a g(A_aX,Y)g(A_at_1V,t_1\mu).$$

Consequently, our equation follows from (1.16).

From (2.2), we have the following

LEMMA 2.2. *The square of the length of A_μ is given by*

$$TrA_\mu^2=\frac{1}{4}c(n-3)+TrA_\mu g(A_\mu t_1\mu,t_1\mu). \tag{2.3}$$

LEMMA 2.3. *The second fundamental forms of M satisfy the following equations:*

$$TrA_\mu TrA_\mu^3-\sum_a(TrA_\mu A_a)^2 \tag{2.4}$$
$$=-\frac{c}{4}nTrA_\mu g(A_\mu t_1\mu,t_1\mu)+\frac{c}{4}(TrA_\mu)^2-\frac{c^2}{16}(n-3)^2.$$

Proof. From (2.2) and $g(A_V t_1 U, t_1 U) = g(A_V t_2 U, t_2 U) = g(A_V t_3 U, t_3 U)$, it follows that

$$TrA_\mu^3 = \frac{c}{4}\{TrA_\mu - 3g(A_\mu t_1\mu, t_1\mu)\} + \sum_a TrA_\mu A_a g(A_\mu t_1\mu, t_1 e_a).$$

Moreover (2.2) implies

$$TrA_\mu A_a = TrA_\mu g(A_\mu t_1\mu, t_1 e_a) + \frac{c}{4}(n-3)g(\mu, e_a),$$

and consequently

$$\begin{aligned}\sum_a (TrA_\mu A_a)^2 =& \frac{c^2}{16}(n-3)^2 + \frac{1}{2}c(n-3)TrA_\mu g(A_\mu t_1\mu, t_1\mu)\\ &+ (TrA_\mu)^2 g(A_\mu t_1\mu, A_\mu t_1\mu),\\ TrA_\mu TrA_\mu^3 =& \frac{c}{4}(TrA_\mu)^2 + \frac{c}{4}(n-6)TrA_\mu g(A_\mu t_1\mu, t_1\mu)\\ &+ (TrA_\mu)^2 g(A_\mu t_1\mu, A_\mu t_1\mu).\end{aligned}$$

From these equations, our equation follows.

By the assumption, we have

$$\sum (\nabla_i A)_\mu e_i = 0,$$

where ∇_i denotes the covariant differentiation in the direction of e_i.

LEMMA 2.4. *The restricted Laplacian for A_μ is given by*

$$\begin{aligned}(\nabla^2 A)_\mu X =& \sum (\nabla_i\nabla_i A)_\mu X &(2.5)\\ =& \sum (R(e_i, X)A)_\mu e_i + \frac{1}{4}c\sum_{r=1}^{3}\{3A_\mu P_r^2 X\\ &- g(F_r X, \mu)\sum A_{F_r e_i} e_i - g(F_r X, \mu)\sum t_r B(e_i, e_i)\\ &- 2\sum g(A_{F_r X} e_i, e_i)t_r\mu - 2\sum g(t_r B(X, e_i), e_i)t_r\mu\}.\end{aligned}$$

Proof. From (1.15), we have

$$(\nabla^2 A)_\mu X = \sum (\nabla_i \nabla_i A)_\mu X$$
$$= \sum (R(e_i, X)A)_\mu e_i + \frac{c}{4} \sum_{r=1}^{3} \sum \{ g((\nabla_i F_r)e_i, \mu) P_r X$$
$$+ g(F_r e_i, \mu)(\nabla_i P_r)X - g((\nabla_i F_r)X, \mu) P_r e_i$$
$$- g(F_r X, \mu)(\nabla_i P_r) e_i - 2g((\nabla_i P_r)X, e_i) t_r \mu$$
$$- 2g(P_r X, e_i)(\nabla_i t_r)\mu \}.$$

Using (1.11), (1.12) and (1.13), we find (2.5).

From (2.5), we have

$$g((\nabla^2 A)_\mu, A_\mu) = \sum g((\nabla_i \nabla_i A)_\mu e_j, A_\mu e_j) \tag{2.6}$$
$$= \sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) + \frac{c}{4} \Big\{ 3 \sum_{r=1}^{3} Tr A_\mu^2 P_r^2$$
$$- 9 \sum_a g(A_a t_1 e_a, A_\mu t_1 \mu) + 9 Tr A_\mu g(A_\mu t_1 \mu, t_1 \mu) \Big\}.$$

On the other hand, by equation of Gauss (1.14), we obtain

$$\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) \tag{2.7}$$
$$= \frac{1}{4} c \{ n Tr A_\mu^2 - (Tr A_\mu)^2 \} + \sum_a Tr (A_\mu A_a)^2$$
$$- \sum_a Tr A_\mu^2 A_a^2 + Tr A_\mu Tr A_\mu^3 - \sum_a (Tr A_\mu A_a)^2.$$

LEMMA 2.5. *The curvature tensor R of M satisfies*

$$\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) = \frac{3}{16} c^2 (n - p - 2). \tag{2.8}$$

Proof. From equation of Ricci (1.16), we have

$$\sum_a Tr (A_\mu A_a)^2 - \sum_a Tr A_\mu^2 A_a^2 = -\frac{3}{16} c^2 (p - 1). \tag{2.9}$$

Substituting (2.3), (2.4) and (2.9) into (2.7), we find (2.8).

LEMMA 2.6. *For the second fundamental form A_μ, we have*

$$g((\nabla^2 A)_\mu, A_\mu) = -\frac{3}{8}c^2(n-p-2). \tag{2.10}$$

Proof. First of all, we have

$$\sum_{r=1}^{3} Tr(A_\mu P_r)^2 = -3TrA_\mu^2 + \sum_{r=1}^{3}\sum_a g(A_a t_r\mu, A_a t_r\mu). \tag{2.11}$$

Furthermore, (2.2) implies

$$\sum_a g(A_a t_1\mu, A_a t_1\mu) = \sum_a g(A_a t_1 e_a, A_\mu t_1\mu) + \frac{1}{4}c(p-1). \tag{2.12}$$

Hence, from (2.3), (2.11) and (2.12), we obtain

$$\begin{aligned} 3\sum_{r=1}^{3} TrA_\mu^2 P_r^2 - 9\sum_a g(A_a t_1 e_a, A_\mu t_1\mu) \\ + 9TrA_\mu g(A_\mu t_1\mu, t_1\mu) = -\frac{9}{4}c(n-p-2). \end{aligned} \tag{2.13}$$

Substituting (2.8) and (2.13) into (2.6), we have (2.10).

3. Theorems

Let M be an n-dimensional generic submanifold of a quaternionic space form $\tilde{M}^{n+p}(c)$ with nonvanishing parallel mean curvature vector. First of all, we prove that $\Delta(TrA_\mu^2) = 0$. From (2.3),

$$\begin{aligned} \Delta(TrA_\mu^2) &= \sum(\nabla_i\nabla_i TrA_\mu^2) \\ &= TrA_\mu\Big\{g((\nabla^2 A)_\mu t_1\mu, t_1\mu) \\ &\quad + 2\sum g((\nabla_i A)_\mu t_1\mu, (\nabla_i t_1)\mu)\Big\}. \end{aligned} \tag{3.1}$$

On the other hand, (2.5) implies

$$g((\nabla^2 A)_\mu t_1\mu, t_1\mu) = \sum g((R(e_i, t_1\mu)A)_\mu e_i, t_1\mu) \\ + \frac{3}{4}c\Big\{TrA_\mu - \sum_a g(A_a t_1 e_a, t_1\mu) \\ - 2g(A_\mu t_1\mu, t_1\mu)\Big\}. \tag{3.2}$$

From (1.12), (1.15) and (1.16), we also have

$$\sum g((\nabla_i A)_\mu t_1\mu, (\nabla_i t_1)\mu) \\ = \frac{c}{4}\Big\{2g(A_\mu t_1\mu, t_1\mu) + \sum_a g(A_a t_1 e_a, t_1\mu) - TrA_\mu\Big\}. \tag{3.3}$$

Using equation of Gauss, we see

$$\sum g((R(e_i, t_1\mu)A)_\mu e_i, t_1\mu) \\ = \sum g(R(e_i, t_1\mu)A_\mu e_i, t_1\mu) - \sum g(R(e_i, t_1\mu)e_i, A_\mu t_1\mu) \\ = -\frac{c}{4}\Big\{TrA_\mu - 3g(A_\mu t_1\mu, t_1\mu)\Big\} + \sum_a g([A_\mu, A_a]t_1\mu, A_a t_1\mu).$$

By the equation of Ricci (1.16), we find

$$\sum g((R(e_i, t_1\mu)A)_\mu e_i, t_1\mu) \\ = \frac{1}{4}c\{\sum_a g(A_a t_1 e_a, t_1\mu) + 2g(A_\mu t_1\mu, t_1\mu) - TrA_\mu\}. \tag{3.4}$$

From (3.1), (3.2), (3.3) and (3.4), we have

LEMMA 3.1. $\Delta(TrA_\mu^2) = 0$.

Hence we have

THEOREM 3.2. *Let M be an n-dimensional generic submanifold of a quaternionic space form $\tilde{M}^{n+p}(c)$ with nonvanishing parallel mean curvature vector. If the second fundamental form of M satisfies the condition $P_rA = AP_r$ $(r = 1, 2, 3)$, then*

$$|(\nabla A)_\mu|^2 = \frac{3}{8}c^2(n-p-2).$$

Proof. Generally, we have

$$\frac{1}{2}\Delta(TrA_\mu^2) = g((\nabla^2 A)_\mu, A_\mu) + |(\nabla A)_\mu|^2 .$$

Thus we have our assertion by Lemma 2.6 and Lemma 3.1.

COROLLARY 3.3. *Under the same assumptions as that of Theorem 3.2, we have*

$$(\nabla_X A)_\mu Y = -\frac{1}{4}c\sum_{r=1}^{3}\{g(F_rY,\mu)P_rX - g(P_rX,Y)t_r\mu\} \tag{3.5}$$

for any vector fields X and Y tangent to M.

Proof. Let us put

$$T(X,Y) = (\nabla_X A)_\mu Y + \frac{1}{4}c\sum_{r=1}^{3}\{g(F_rY,\mu)P_rX - g(P_rX,Y)t_r\mu\}.$$

Then we have, by equation of Codazzi (1.15),

$$|T|^2 = |(\nabla A)_\mu|^2 - \frac{3}{8}c^2(n-p-2) \geq 0.$$

Therefore, T vanishes identically if and only if

$$|(\nabla A)_\mu|^2 = \frac{3}{8}c^2(n-p-2).$$

THEOREM 3.4. *Let M be an n-dimensional generic submanifold of a quaternionic space form $\tilde{M}^{4m}(c)$ with nonvanishing parallel mean curvature vector. If $P_rA = AP_r$ $(r = 1,2,3)$, and if the sectional curvature of M is nonpositive, then the second fundamental form in the direction of the mean curvature vector is parallel. So, $c = 0$ or M is real hypersurface in $\tilde{M}^4(c)$.*

Proof. We take an orthonormal basis $e_1, \ldots, e_n$ such that $A_\mu e_i = \lambda_i e_i$. We denote by K_{ij} the sectional curvature of M spanned by e_i, e_j. Then we have

$$\sum g((R(e_i,e_j)A)_\mu e_i, A_\mu e_j) = \frac{1}{2}\sum(\lambda_i - \lambda_j)^2 K_{ij}.$$

Substituting this into (2.8), we obtain

$$\sum(\lambda_i - \lambda_j)^2 K_{ij} = \frac{3}{8}c^2(n-p-2) \geq 0.$$

Thus if $K_{ij} \leq 0$, then $c^2(n-p-2) = 0$ and hence $(\nabla A)_\mu = 0$ by Theorem 3.2. Moreover, we have $c = 0$ or $(n-p-2) = 0$. If $n-p-2 = 0$, then since M is generic submanifold of $\tilde{M}^{n+p}(c)$, $p+2 \geq 3p$. i.e., $2(1-p) \geq 0$. Hence p=0 or 1.

If $p = 1$, $n = 3$, and consequently M is a real hypersurface in $\tilde{M}^4(c)$. If $p = 0$, then $n = 2$. This is a contradiction.

REMARK. Theorem 3.4 implies that, under the assumption of Theorem 3.4, the ambient space $\tilde{M}^{4m}(c)$ admitting any generic submanifold with nonpositive sectional curvature is only Euclidean space provided $4m \geq 8$.

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