

# MANN-ITERATION PROCESS FOR THE FIXED POINT OF STRICTLY PSEUDOCONTRACTIVE MAPPING IN SOME BANACH SPACES

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## 1. Introduction

Many authors[3][4][5] constructed and examined some processes for the fixed point of strictly pseudocontractive mapping in various Banach spaces. In fact the fixed point of strictly pseudocontractive mapping is the zero of strongly accretive operators. So the same processes are used for the both circumstances. Reich[3] proved that Mann-iteration process can be applied to approximate the zero of strongly accretive operator in uniformly smooth Banach spaces. In the above paper he asked whether the fact can be extended to other Banach spaces the duals of which are not necessarily uniformly convex. Recently Schu[4] proved it for uniformly continuous strictly pseudocontractive mappings in smooth Banach spaces. In this paper we proved that Mann-iteration process can be applied to approximate the fixed point of strictly pseudocontractive mapping in certain Banach spaces.

## 2. Main result

Let  $(X, \|\cdot\|)$  be a Banach space. A Banach space  $(X, \|\cdot\|)$  is called smooth if the norm of  $X$  is Gâteaux differentiable on  $X - \{0\}$ . The duality map  $J$  is defined by

$$J(x) = \{ x^* \in X^* \mid (J(x), x) = \|x\|^2, \|J(x)\| = \|x\| \},$$

where  $X^*$  is the dual of  $X$  and  $(\cdot, \cdot)$  is the dual pairing. In a smooth Banach space  $J$  is single-valued. A Banach space  $(X, \|\cdot\|)$  is called uniformly smooth if  $X^*$ (the dual of  $X$ ) is uniformly convex. In a

uniformly smooth Banach space  $J$  is uniformly continuous on bounded subsets of  $X$ . Let  $C$  be a nonempty subset of a smooth Banach space. A map  $T : C \rightarrow X$  is said to be strictly pseudocontractive if  $T$  satisfies the following condition:

$$(J(x - y), Tx - Ty) \leq \gamma \|x - y\|^2$$

for all  $x, y \in C$  and some  $\gamma \in [0, 1)$ . If  $T$  is strictly pseudocontractive, then  $F = Id - T$  is strongly accretive, i.e.,

$$(J(x - y), Fx - Fy) \geq (1 - \gamma) \|x - y\|^2$$

for all  $x, y \in C$ . Furthermore if  $T$  is an contraction of  $C$ , then  $T$  is strictly pseudocontractive. It is known [2, Corollary 1] that a continuous strictly pseudocontractive selfmapping of a closed convex subset of a Banach space has the unique fixed point.

The following iterative construction is called Mann-iteration process:

- (1)  $x_0 \in C$
- (2)  $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n, \alpha_n \in (0, 1)$ .

In case  $X$  is  $L_p$  or  $l_p$  with  $p \in [2, \infty)$ , it was shown [1] that Mann-iteration process converges strongly to the fixed point of a strictly pseudocontractive mapping  $T$ , provided that  $T$  is additionally Lipschitzian and  $\{\alpha_n\}$  satisfies  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ . Schu[4] generalized the result in [1] in smooth Banach space. Weng[5] also proved it under the assumption that  $T$  is not necessarily continuous and  $T$  has a fixed point and  $\{\alpha_n\}$  satisfies  $\lim_{n \rightarrow \infty} \alpha_n = 0$ .

We obtain the following lemma which is similar to the lemma of Weng[5].

**LEMMA 1.** *Let  $\{\beta_n\}$  be a nonnegative real sequence and suppose  $\{\beta_n\}$  satisfies the following inequality*

$$\beta_{n+1} \leq (1 - \alpha_n) \beta_n + \varepsilon \alpha_n,$$

where  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\alpha_n \in (0, 1)$ ,  $\varepsilon > 0$ . Then

$$0 \leq \limsup_{n \rightarrow \infty} \beta_n \leq \varepsilon.$$

*Proof.* By the mathematical induction

$$\beta_{n+1} \leq (1 - \alpha_n)(1 - \alpha_{n-1}) \cdots (1 - \alpha_1)\beta_1 + \varepsilon. \quad (2.1)$$

Indeed for  $n = 1$ , we have  $\beta_2 \leq (1 - \alpha_1)\beta_1 + \varepsilon\alpha_1 \leq (1 - \alpha_1)\beta_1 + \varepsilon$ . Suppose the lemma holds for  $n$ . Then

$$\begin{aligned} \beta_{n+2} &\leq (1 - \alpha_{n+1})\beta_{n+1} + \varepsilon\alpha_{n+1} \\ &\leq (1 - \alpha_{n+1})((1 - \alpha_n)(1 - \alpha_{n-1}) \cdots (1 - \alpha_1)\beta_1 + \varepsilon) + \varepsilon\alpha_{n+1} \\ &\leq (1 - \alpha_{n+1})(1 - \alpha_n) \cdots (1 - \alpha_1)\beta_1 + (1 - \alpha_{n+1})\varepsilon + \alpha_{n+1}\varepsilon \\ &\leq (1 - \alpha_{n+1})(1 - \alpha_n) \cdots (1 - \alpha_1)\beta_1 + \varepsilon. \end{aligned}$$

Since  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , we have the conclusion from (2.1).  $\square$

On the other hand, we need the following lemma.

**LEMMA 2.** *Let  $(X, \|\cdot\|)$  be a smooth Banach space. of  $X$ . Suppose one of the followings holds.*

- (1)  $J$  is uniformly continuous on any bounded subsets of  $X$ .
- (2)  $(J(x) - J(y), x - y) \leq \|x - y\|^2$ , for all  $x, y$  in  $X$ .
- (3) For any bounded subset  $D$  of  $X$  there is a  $c$  such that

$$(J(x) - J(y), x - y) \leq c(\|x - y\|),$$

for all  $x, y$  in  $D$  where  $c$  satisfies  $\lim_{t \rightarrow 0^+} c(t)/t = 0$ .

Then for any  $\varepsilon > 0$  and any bounded subset  $C$  there is  $\delta > 0$  such that

$$\|tx + (1 - t)y\|^2 \leq 2(J(y), x)t + 2\varepsilon t + (1 - 2t)\|y\|^2,$$

for any  $x, y \in C$  and  $t \in [0, \delta)$ .

*Proof.* Since (2) implies (3), we will prove the lemma under the hypothesis(3). The proof of the lemma under (1) is analogous to the proof given here. For the convex hull of  $C$  we choose  $c$  and  $\varepsilon > 0$  be arbitrarily given. Since  $X$  is a smooth Banach space,

$$\frac{1}{2} \frac{d}{dt} \|tx + (1 - t)y\|^2 = (J(tx + (1 - t)y), x - y), x, y \in X. \quad (2-2)$$

Since  $\lim_{t \rightarrow 0^+} c(t)/t = 0$ , there is  $\delta' > 0$  such that for  $t \in [0, \delta')$ ,

$$c(t)/t \leq \frac{\varepsilon}{\text{diam } C}.$$

Hence for any  $x, y \in C$

$$\begin{aligned} & (J(tx + (1 - t)y), t(x - y)) - (J(y), t(x - y)) \\ &= (J(tx + (1 - t)y) - J(y), t(x - y)) \\ &\leq c(t)\|x - y\|. \end{aligned}$$

Therefore

$$\begin{aligned} & (J(tx + (1 - t)y) - J(y), x - y) \\ &\leq c(t)\|x - y\|/t \\ &= (c(t)\|x - y\|)/t \cdot \|x - y\|\|x - y\| \\ &\leq (c(t)\|x - y\|)/t\|x - y\|\text{diam } C. \end{aligned} \tag{2.3}$$

Let  $\delta = \delta'/\text{diam } C$ . From (2-3), we obtain

$$(J(tx + (1 - t)y) - J(y), x - y) < \varepsilon$$

, for any  $t \in [0, \delta)$ . So  $(J(tx + (1 - t)y), x - y) \leq (J(y), x - y) + \varepsilon$ . From (2-2) we have

$$\frac{1}{2} \frac{d}{dt} \|tx + (1 - t)y\|^2 \leq (J(y), x - y) + \varepsilon.$$

If we integrate the above inequality from 0 to  $t \in [0, \delta)$ , we have the conclusion.  $\square$

REMARK. If  $X$  is uniformly smooth, then (1) in lemma 2 holds. And if  $X$  is a Hilbert space, then (2) in lemma 2 holds.

Now we state the main result.

THEOREM. Let  $(X, \|\cdot\|)$  be a smooth Banach space under one of the assumptions in Lemma 2 and  $C$  a closed bounded convex subset of  $X$ . Let  $T : C \rightarrow C$  be a continuous strictly pseudocontractive mapping with  $\gamma \in [0, 1)$ . Then Mann-iteration process:

- (1)  $x_1 \in C$
- (2)  $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n)x_n, \alpha_n \in (0, 1)$
- (3)  $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \rightarrow \infty} \alpha_n = 0$

converges strongly to the fixed point of  $T$ .

*Proof.* By [2, Corollary 1],  $T$  has a unique fixed point  $x$  in  $C$ . For given  $\varepsilon > 0$  and a bounded subset  $C - C$ , we have  $\delta > 0$  in Lemma 2. Since  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , there exists  $N$  such that for all  $n \geq N$ ,  $\alpha_n < \delta$ . For such an  $n$ ,

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|\alpha_n(Tx_n - x) + (1 - \alpha_n)(x_n - x)\|^2 \\ &\leq 2(J(x_n - x), Tx_n - x)\alpha_n + 2\varepsilon\alpha_n + (1 - 2\alpha_n)\|x_n - x\|^2 \end{aligned}$$

by Lemma 2.

Since  $T$  is a strictly pseudocontractive mapping with  $\gamma \in [0, 1)$  and  $x$  is the fixed point of  $T$ , the following holds.

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq 2\gamma\|x_n - x\|^2\alpha_n + 2\varepsilon\alpha_n + (1 - 2\alpha_n)\|x_n - x\|^2 \\ &= (1 - 2(1 - \gamma)\alpha_n)\|x_n - x\|^2 + 2\varepsilon\alpha_n. \end{aligned}$$

In order to apply Lemma 1 we let  $\beta_n = \|x_n - x\|^2$ . Then

$$0 \leq \limsup_{n \rightarrow \infty} \beta_n \leq \varepsilon$$

. Since  $\varepsilon$  is arbitrary,  $\limsup_{n \rightarrow \infty} \beta_n = 0$ . So  $\lim_{n \rightarrow \infty} \beta_n = 0$ . The proof is complete.  $\square$

We remark that Theorem holds without the continuity of  $T$  if  $T$  has the fixed point in  $C$ .

## References

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