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THE PEETRE K–METHOD IN STRONG λ–K–MONOTONE SPACES

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I. Introduction

A Banach couple $A = (A_0, A_1)$ is a pair of Banach spaces A_0 and A_1 which are algebraically and topologically imbedded into a Hausdorff topological vector space Γ. For a Banach couple $\bar{A} = (A_0, A_1)$, the intersection $A_0 \cap A_1$ and the sum $A_0 + A_1$ are defined with the natural norms $J(1, a, \bar{A})$ and $K(1, a, \bar{A})$ respectively, where

$$J(t, a, \bar{A}) = \max\,(\|a\|_{A_0}, t\|a\|_{A_1}), \quad a \in A_0 \cap A_1$$
$$K(t, a, \bar{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1 \text{ for } t \in (0, \infty).$$

A is called an intermediate space with respect to $\bar{A} = (A_0, A_1)$ if $A_0 \cap A_1 \subset A \subset A_0 + A_1$. As usual, the inclusion map from A to B is denoted by $A \subset B$ and assumed to be continuous. For Banach couple $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$, let $L(\bar{A}, \bar{B})$ denote the set of all linear operators T such that T maps A_i into B_i boundedly for $i = 0, 1$. We define the norm of $T \in L(\bar{A}, \bar{B})$ as

$$\|T\|_{\bar{A},\bar{B}} = \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}).$$

Then $L(\bar{A}, \bar{B})$ is a Banach space with the above norm. If A and B are intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively and for every $T \in L(\bar{A}, \bar{B})$, T is bounded from A to B, we say A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. In this case, the open mapping theorem gives that

$$\|T\|_{A,B} \le C\|T\|_{\bar{A},\bar{B}}, \tag{1-1}$$

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where C is independent of T (see theorem 2.4.2 in [2]).

There are some methods which allow us to construct interpolation spaces A and B with respect to given Banach couples $\bar{A}$ and $\bar{B}$. We introduce the Peetre K-method and the Aronszajn-Gagliardo method. We modify their statements for our use and provide proofs. In the next section, we shall show that these methods are quite general, i.e., for some Banach couples, every interpolation space can be constructed by using these methods. The following simple fact will be used frequently. For Banach couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, let $a \in A_0 + A_1$ and $T \in L(\bar{A}, \bar{B})$. Then we have

$$K(t, Ta, \bar{B}) \leq \|T\|_{\bar{A},\bar{B}} K(t, a, \bar{A}), \tag{1-2}$$

since, for $a = a_0 + a_1$,

$$\begin{aligned} K(t, Ta, \bar{B}) &\leq \|Ta_0\|_{B_0} + t\|Ta_1\|_{B_1} \\ &\leq \|T\|_{\bar{A},\bar{B}}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \end{aligned}$$

and the above inequality holds for any representation of a. For notation, let $[a_n]_{-\infty}^{\infty}$ denote a sequence on the set of integers.

THEOREM 1. *(theorem 3.1 in [6]). Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples. Let Ψ be a lattice sequence norm satisfying*

$$0 < \Psi([\min(1, 2^n)]_{-\infty}^{\infty}) < \infty.$$

Let

$$\bar{A}_\Psi = \{a \in A_0 + A_1; \Psi([K(2^n, a, \bar{A})]_{-\infty}^{\infty}) < \infty\}$$

with the norm $\|a\|_{\bar{A}_\Psi} = \Psi([K(2^n, a, \bar{A})]_{-\infty}^{\infty})$, for $a \in \bar{A}_\Psi$, and

$$\bar{B}_\Psi = \{b \in B_0 + B_1; \Psi([K(2^n, b, \bar{B})]_{-\infty}^{\infty}) < \infty\}$$

with the norm $\|b\|_{\bar{B}_\Psi} = \Psi([K(2^n, b, \bar{B})]_{-\infty}^{\infty})$ for $b \in \bar{B}_\Psi$. Then $\bar{A}_\Psi$ and $\bar{B}_\Psi$ are interpolation space with respect to $\bar{A}$ and $\bar{B}$.

Proof. Since Ψ is a lattice norm, it is clear that $\|\cdot\|_{\bar{A}_\Psi}$ and $\|\cdot\|_{\bar{B}_\Psi}$ are norms. First, we show that $\bar{A}_\Psi$ is an intermediate space with respect to $\bar{A}$. Let $\Psi([\min(1,2^n)]_{-\infty}^{\infty}) = \alpha$. For the inclusion $A_0 \cap A_1 \subset \bar{A}_\Psi$, note that

$$K(2^n, a, \bar{A}) \leq \min(1,2^n)\|a\|_{A_0\cap A_1}$$

for $a \in A_0 \cap A_1$ and for all integers n.

Thus,

$$\|a\|_{\bar{A}_\Psi} \leq \alpha\|a\|_{A_0\cap A_1}.$$

For $\bar{A}_\Psi \subset A_0 + A_1$, observe the following simple fact

$$\min(1,2^n)K(1,a,\bar{A}) \leq K(2^n,a,\bar{A}),$$

for $a \in A_0 + A_1$. By taking the lattice norm Ψ on the above inequality, we get

$$\alpha\|a\|_{A_0+A_1} \leq \|a\|_{\bar{A}_\Psi}.$$

By the same way, we can show $\bar{B}_\Psi$ is also an intermediate space with respect to $\bar{B}$. Suppose $T \in L(\bar{A},\bar{B})$ and $a \in \bar{A}_\Psi$. Then, formula 1-2 gives

$$K(2^n, Ta, \bar{B}) \leq \|T\|_{\bar{A},\bar{B}}K(2^n,a,\bar{A}),$$

for all n. Thus,

$$\|Ta\|_{\bar{B}_\Psi} \leq \|T\|_{\bar{A},\bar{B}}\|a\|_{\bar{A}_\Psi}.$$

This implies that T is bounded from $\bar{A}_\Psi$ to $\bar{B}_\Psi$. ■

THEOREM 2. *(theorem 11.1 in [1]). Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples and let A be an intermediate space with respect to $\bar{A}$. Then there exists an intermediate space R_A with respect to $\bar{B}$ such that A and R_A are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. Futhermore, if A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$, we have*

$$R_A \subset B.$$

Proof. Define R_A to be the set which consists of those $y \in B_0 + B_1$ such that y has a representation $y = \sum_{i=1}^{\infty} T_i a_i$ (convergence in B_0+B_1) and $\sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}}\|a_i\|_A < \infty$, where $T_i \in L(\bar{A},\bar{B})$ and $a_i \in A$. For $y \in R_A$, we define the norm of y as

$$\|y\|_{R_A} = \inf_{y=\Sigma T_i a_i} \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}}\|a_i\|_A.$$

Then R_A is a Banach space since $\bar{B}$ is a Banach couple. By the construction of the space R_A, it is clear that T is bounded from A to R_A with $\|T\|_{A,R_A} \leq \|T\|_{\bar{A},\bar{B}}$, if $T \in L(\bar{A},\bar{B})$. Now we show that R_A is an intermediate space with respect to $\bar{B}$. Let $b \in B_0 \cap B_1$. Let $a^* \in A_0 \cap A_1$ be fixed and ψ be a linear functional on $A_0 + A_1$ such that $\psi(a^*) = 1$. Define, for $b \in B_0 \cap B_1$,

$$T(x) = \psi(x)b$$

from $A_0 + A_1$ to $B_0 + B_1$. Since $b \in B_0 \cap B_1$, we have, for $x \in A_i$ and $i = 0, 1$,

$$\begin{aligned}
\|Tx\|_{B_i} &= |\psi(x)|\|b\|_{B_i} \\
&\leq \|\psi\|_{A_0+A_1}\|x\|_{A_0+A_1}\|b\|_{B_i} \\
&\leq \|\psi\|_{A_0+A_1}\|b\|_{B_i}\|x\|_{A_i}.
\end{aligned}$$

The last inequality is obtained from $A_i \subset A_0 + A_1$. This implies that T is in $L(\bar{A},\bar{B})$ with $\|T\| \leq \|\psi\|_{A_0+A_1}\|b\|_{B_0\cap B_1}$. Since $Ta^* = b$,

$$\begin{aligned}
\|b\|_{R_A} &\leq \|T\|_{\bar{A},\bar{B}}\|a^*\|_A \\
&\leq \|\psi\|_{A_0+A_1}\|a^*\|_A\|b\|_{B_0\cap B_1}.
\end{aligned}$$

Thus we have $B_0 \cap B_1 \subset R_A$. For the other inclusion, let $y = \sum_{i=1}^{\infty} T_i a_i$ be a representation of y in R_A.

Since

$$\begin{aligned}
\|y\|_{B_0+B_1} &= \|\sum_{i=1}^{\infty} T_i a_i\|_{B_0+B_1} \\
&\leq \sum_{i=1}^{\infty} \|T_i a_i\|_{B_0+B_1} \\
&\leq \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}}\|a_i\|_{A_0+A_1} \\
&\leq C\sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}}\|a_i\|_A,
\end{aligned}$$

and the above inequality holds for every representation of y, we have

$$\|y\|_{B_0+B_1} \leq C\|y\|_{R_A}.$$

Suppose that A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. Let y be in R_A with $y = \sum_{i=1}^{\infty} T_i a_i$. Then,

$$\begin{aligned}\|y\|_B &\leq \sum_{i=1}^{\infty} \|T_i a_i\|_B \leq \sum_{i=1}^{\infty} \|T_i\|_{A,B}\|a_i\|_A \\ &\leq C \sum \|T_i\|_{\bar{A},\bar{B}}\|a_i\|_A \quad \text{by (1-1).}\end{aligned}$$

Thus we have $R_A \subset B$.■

2. Main results

In [5], M. Cwikel and J. Peetre showed that some Banach couples are Calderon pairs and every K-monotone cones in that couples can be constructed by the Peetre K-method. With the help of theorem 4 in [3], we can extend the above result to the K-monotone spaces. In [4], M. Cwikel provided the proof of theorem 4 in [3] and also mentioned that some theorems in [5] which were stated in terms of K-monotone cones can be extended to K-monotone spaces. But M. Cwikel and J. Peetre's results only cover the case when $\bar{A} = \bar{B}$. We extend this result to the case when $\bar{A} \neq \bar{B}$. In this case, we need a condition stronger than that of the K-monotone space.

DEFINITION. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples. An intermediate space A with respect to $\bar{A}$ is a strong λ-K-monotone space with respect to $\bar{A}$ and $\bar{B}$, if, for $a \in A$ and $b \in B_0 + B_1$,

$$K(t, b, \bar{B}) \leq K(t, a, \bar{A}),$$

for all $t > 0$, there exists a $T \in L(\bar{A}, \bar{B})$ such that $b = Ta$ and $\|T\|_{\bar{A},\bar{B}} \leq \lambda$.

For the examples of λ-K-monotone space, see [6]. When $\bar{A} = (L_{p_0}, L_{p_1})\bar{B} = (L_{q_0}, L_{q_1})$, every intermediate space A with respect to $\bar{A}$ is also a strong λ-K-monotone space.

THEOREM 3. *(theorem 4 in [3] and theorem 1 in [4]).*
Let $\bar{B} = (B_0, B_1)$ be a Banach couple and let b be an element of $B_0 + B_1$. Suppose that $K(t, b, \bar{B}) \leq \sum_{i=1}^{\infty} \Psi_i(t)$, for all positive numbers t, where each $\Psi_i(t)$ is a positive concave function on $(0, \infty)$ and $\sum_{i=1}^{\infty} \Psi_i(1) < \infty$. Then there exist elements $b_i \in B_0 + B_1$ and a positive number r such that $b = \sum_{i=1}^{\infty} b_i$ (convergence in $B_0 + B_1$) and $K(t, b_i, \bar{B}) \leq r\Psi_i(t)$, for each i and all positive numbers t.

COROLLARY. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples. Let A be a strong λ-K-monotone space with respect to $\bar{A}$ and $\bar{B}$. If, for $b \in B_0 + B_1$ and $a_i \in A_0 + A_1$, $i = 1, \cdots, N$, the inequality*

$$K(2^n, b, \bar{B}) \leq \sum_{i=1} NK(2^n, a_i, \bar{A}) \tag{2-1}$$

holds for all integers n, then there exist $T_i \in L(\bar{A}, \bar{B})$ and a positive number r such that

$$b = \sum_{i=1} NT_i a_i, \quad \|T_i\|_{\bar{A},\bar{B}} \leq \lambda$$

and

$$K(2^n, T_i a_i, \bar{B}) \leq rK(2^n, a_i, \bar{A}).$$

Futhermore, we have

$$\|b\|_{R_A} \leq 2r\lambda \sum_{i=1} N\|a_i\|_A. \tag{2-2}$$

Proof. For $t > 0$, there exists an integer n such that $2^n \leq t < 2^{n+1}$. Then, (2-1) implies that, for $t > 0$,

$$\begin{aligned} K(t, b, \bar{B}) &\leq 2K(2^n, b, \bar{B}) \\ &\leq 2\sum_{i=1} NK(2^n, a_i, \bar{A}) \\ &\leq 2\sum_{i=1} NK(t, a_i, \bar{A}) \\ &= \sum_{i=1} NK(t, 2a_i, \bar{A}). \end{aligned}$$

Since $K(t, a, \bar{A})$ is a positive concave function (lemma 2.3.1 in [2]), there exist $b_i \in B_0 + B_1$ and a positive number r such that $b = \sum_{i=1} N b_i$ and

$$K(2^n, b_i, \bar{B}) \leq rK(2^n, 2a_i, \bar{A})$$

for each i and all integers n by theorem 3. Since A is a strong λ-K-monotone space, we get $T_i \in L(\bar{A}, \bar{B})$ such that

$$b_i = T_i(2ra_i), \;\; b = \sum_{i=1} NT_i(2ra_i) \;\text{ and }\; \|T_i\|_{\bar{A},\bar{B}} \leq \lambda.$$

Thus

$$\begin{aligned}\|b\|_{R_A} \leq & \sum_{i=1} N\|T_i\|_{\bar{A},\bar{B}}\|2ra_i\|_A \\ \leq & 2r\lambda \sum_{i=1} N\|a_i\|_A. \blacksquare\end{aligned}$$

For notations, let $[\alpha_n], [\alpha_n^i]$ denote sequences on the set of $\{n; n$ is an integer $\}$ and α_n, α_n^i denote the nth coordinate of sequences $[\alpha_n]$, $[\alpha_n^i]$ respectively.

LEMMA 1. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let A be an intermediate space with respect to $\bar{A}$. Define Φ, Ψ on the set of sequences $[\alpha_n]$ as*

$$\begin{aligned}\Phi([\alpha_n]) =& \inf\{\|a\|_A;\; |\alpha_n| \leq K(2^n, a, \bar{A}) \;\; \textit{for all } n\} \;\; \textit{and} \\ \Psi([\alpha_n]) =& \inf\{\sum_{i=1} k\Phi([\alpha_n^i]);\; \alpha_n = \sum_{i=1} k\alpha_n^i \;\; \textit{for all } n\},\end{aligned}$$

where the second infimum is taken over all sequences $[\alpha_n^i]$ such that the sequence $[\alpha_n]$ is the finite sum of $[\alpha_n^i]$. Then Ψ is a lattice sequence norm and satisfies

$$0 < \Psi([\min(1, 2^n)]_{-\infty}^{\infty}) < \infty.$$

Proof. $\Psi([r\alpha_n]) = |r|\Psi([\alpha_n])$ is clear. For $\epsilon > 0$, There exist $[\alpha_n^i]$ and $[\beta_n^i]$ such that $[\alpha_n] = \sum_{i=1}^{s} \Phi([\alpha_n^i])$, $[\beta_n] = \sum_{i=1}^{t} \Phi([\beta_n^i])$ and

$$\sum_{i=1}^{s} \Phi([\alpha_n^i]) \leq \Psi([\alpha_n]) + \frac{\epsilon}{2}$$

$$\sum_{i=1}^{t} \Phi([\beta_n^i]) \leq \Psi([\beta_n]) + \frac{\epsilon}{2}.$$

Since the nth coordinate of the sequence $[\alpha_n] + [\beta_n]$ is $\sum_{i=1} s\alpha_n^i + \sum_{i=1} t\beta_n^i$,

$$\begin{aligned}\Psi([\alpha_n + \beta_n]) \leq& \sum_{i=1}^{s} \Phi([\alpha_n^i]) + \sum_{i=1}^{t} \Phi([\beta_n^i]) \\ \leq& \Psi([\alpha_n]) + \Psi([\beta_n]) + \epsilon.\end{aligned}$$

This shows the triangular inequality of Ψ. If $|\alpha_n| \leq |\beta_n|$, $\Phi([\alpha_n]) \leq \Phi([\beta_n])$. For $\epsilon > 0$, there exist sequences $[\beta_n^i]$ such that $[\beta_n] = \sum_{i=1}^{N} [\beta_n^i]$ and $\sum_{i=1}^{N} \Phi([\beta_n^i]) \leq \Psi([\beta_n]) + \epsilon$. We may assume β_n, β_n^i and α_n are all positive sequences. Since $\alpha_n \leq \beta_n$, there exist sequences $[\alpha_n^i]$ such that $[\alpha_n] = \sum_{i=1}^{N} [\alpha_n^i]$ and $\alpha_n^i \leq \beta_n^i$ for $i = 1, 2, \cdots, N$. Thus,

$$\Psi([\alpha_n]) \leq \sum_{i=1}^{N} \Phi([\alpha_n^i]) \leq \sum_{i=1}^{N} \Phi([\beta_n^i]) \leq \Psi([\beta_n]) + \epsilon.$$

Suppose $\Psi([\alpha_n]) = 0$ for $[\alpha_n] \neq 0$. We may assume $\alpha_n \geq 0$, and so there exists an integer s such that $\alpha_s > 0$. Let $M = \max(1, 2^s)$. Since $A \subset A_0 + A_1$, there exists a constant C such that

$$\|a\|_{A_0+A_1} \leq C\|a\|_A. \tag{2-3}$$

Take ϵ such that $0 < \epsilon < \frac{\alpha_s}{2}$. Then, for $\frac{\epsilon}{2CM}$, there exist sequences $[\alpha_n^i]$, for $i = 1, 2, \cdots, N$, such that $[\alpha_n] = \sum_{i=1} N[\alpha_n^i]$ and

$$0 = \Psi([\alpha_n]) \leq \sum_{i=1}^{N} \Phi([\alpha_n^i]) \leq \frac{\epsilon}{2CM} \tag{2-4}$$

By the definition of $\Phi([\alpha_n^i])$, for $\epsilon_i > 0$ with $\sum_{i=1}^N \epsilon_i \le \frac{\epsilon}{2CM}$, there exist $a_i \in A$ such that

$$\alpha_n^i \le K(2^n, a_i, \bar{A}) \quad \text{for all } n \text{ and}$$

$$\|a_i\|_A \le \Phi([\alpha_n^i]) + \epsilon_i. \tag{2-5}$$

In particular, we have $\alpha_s^i \le K(2^s, a_i, \bar{A})$. Thus, for $i = 1, 2, \cdots, N$,

$$\begin{aligned} \alpha_s^i \le K(2^s, a_i, \bar{A}) &\le \max\,(1, 2^s) K(1, a_i, \bar{A}) \\ &\le C \max\,(1, 2^s) \|a_i\|_A \quad \text{by (2-3)} \\ &\le CM\{\Phi([\alpha_n^i]) + \epsilon_i\} \quad \text{by (2-5)} \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_s = \sum_{i=1}^N \alpha_s^i &\le \sum_{i=1}^N CM\{\Phi([\alpha_n^i]) + \epsilon_i\} \\ &\le CM\left(\frac{\epsilon}{2CM} + \frac{\epsilon}{2CM}\right) \quad \text{by } (2-4) \\ &\le \epsilon < \frac{\alpha_s}{2}. \end{aligned}$$

This is a contradiction. Thus, it shows that Ψ is a lattice norm. Finally, we will show $\Psi([\min(1, 2^n)]_{-\infty}^{\infty}) < \infty$. Take $a \in A$ such that $K(1, a, \bar{A}) = 1$. Since, for all integers n,

$$\min(1, 2^n) = \min(1, 2^n) K(1, a, \bar{A}) \le K(2^n, a, \bar{A}),$$

we have $\Phi([\min(1, 2^n)]) \le \|a\|_A$ and so $\Psi([\min(1, 2^n)]) \le \|a\|_A$. ■

LEMMA 2. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples and let A be an intermediate space with respect to $\bar{A}$. Let R_A be the space defined in theorem 2 with respect to $\bar{A}$ and $\bar{B}$. If there is a lattice sequence norm Ψ such that $A \subset \bar{A}_\Psi$, then $R_A \subset \bar{B}_\Psi$.*

Proof. Let $y \in R_A$. Then there is a representation of y such that $y = \sum_{i=1}^\infty T_i a_i$, where $T_i \in L(\bar{A}, \bar{B})$ and $a_i \in A$. Since the K-functional satisfies the triangular inequality,

$$\begin{aligned} K(2^n, y, \bar{B}) =& K\Big(2^n, \sum_{i=1}^{\infty} T_i a_i, \bar{B}\Big) \\ \leq& \sum_{i=1}^{\infty} K(2^n, T_i a_i, \bar{B}) \\ \leq& \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}} K(2^n, a_i, \bar{A}) \quad \text{by (1-2)}, \end{aligned}$$

for all n. Thus,

$$\begin{aligned} \Psi([K(2^n, y, \bar{B})]_{-\infty}^{\infty}) \leq& \sum_{i=1}^{\infty} \Psi([\|T_i\|_{\bar{A},\bar{B}} K(2^n, a_i, \bar{A})]_{-\infty}^{\infty}) \\ =& \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}} \Psi([K(2^n, a_i, \bar{A})]_{-\infty}^{\infty} \\ =& \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}} \|a_i\|_{\bar{A}_\Psi} \\ \leq& C \sum_{i=1}^{\infty} \|T_i\|_{\bar{A},\bar{B}} \|a_i\|_{A}, \end{aligned}$$

since $A \subset \bar{A}_\Psi$.

Since the above inequality holds for every representation of y as $\sum_{i=1}^{\infty} T_i a_i$, we have

$$\|y\|_{\bar{B}_\Psi} \leq C\|y\|_{R_A}.$$

Thus $$R_A \subset \bar{B}_\Psi. \qquad \blacksquare$$

THEOREM 4. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be given Banach couples. If A is a strong λ-K-monotone space with respect to $\bar{A}$ and $\bar{B}$, then there exists a lattice sequence norm Ψ such that $A \subset \bar{A}_\Psi$ and $R_A = \bar{B}_\Psi$. Futhermore, if A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$, then we have $R_A \subset B$.*

Proof. Define Ψ as the lattice sequence norm in lemma 1. First, we show that $A \subset \bar{A}_\Psi$. Since $\Phi([K(2^n, a, \bar{A})]_{-\infty}^{\infty}) \le \|a\|_A$ by the definition of Φ,

$$\|a\|_{\bar{A}_\Psi} \le \Phi([K(2^n, a, \bar{A})]_{-\infty}^{\infty}) \le \|a\|_A.$$

The inclusion $R_A \subset \bar{B}_\Psi$ is clear by lemma 2. For $\bar{B}_\Psi \subset R_A$, let $y \in \bar{B}_\Psi$. Since $\Psi([K(2^n, y, \bar{B})]_{-\infty}^{\infty}) < \infty$, for $\epsilon > 0$, there exist $[\beta_n^i]$ $i = 1, \cdots, N$ such that

$$[K(2^n, y, \bar{B})] = \sum_{i=1}^{N} [\beta_n^i]$$

and

$$\sum_{i=1}^{N} \Phi([\beta_n^i]) \le \Psi([K(2^n, y, \bar{B})] + \frac{\epsilon}{4\lambda r}, \tag{2-6}$$

where r is the constant chosen in theorem 3. Since, for each $[\beta_n^i]$, there exists a_i in A such that

$$[\beta_n^i] \le [K(2^n, a_i, \bar{A})] \quad \text{and}$$

$$\|a_i\|_A \le \Phi([\beta_n^i]) + \frac{\epsilon}{4\lambda r N}, \tag{2-7}$$

we have

$$[K(2^n, y, \bar{B})] = \sum_{i=1}^{N} [\beta_n^i] \le \sum_{i=1}^{N} [K(2^n, a_i, \bar{A})].$$

By the corollary of theorem 3, the above inequality means that there exist $T_i \in L(\bar{A}, \bar{B})$ with $\|T_i\|_{\bar{A},\bar{B}} \le \lambda$ for $i = 1, \cdots, N$, and

$$y = \sum_{i=1}^{N} T_i a_i \quad \text{and}$$

$$\|y\|_{R_A} \le 2r\lambda \sum_{i=1}^{N} \|a_i\|_A.$$

Thus

$$\begin{aligned}\|y\|_{R_A} &\leq 2r\lambda\left\{\sum_{i=1}^{N}\left\{\Phi([\beta_n^i]) + \frac{\epsilon}{4\lambda N}\right\}\right\} \quad \text{by } (2-7)\\ &\leq 2r\lambda\left\{\Psi([K(2^n, y, \bar{B})] + \frac{\epsilon}{4\lambda r}\right\} + \frac{\epsilon}{2} \quad \text{by } (2-6)\\ &\leq 2r\lambda\|y\|_{\bar{B}_\Psi} + \epsilon.\end{aligned}$$

Therefore $\bar{B}_\Psi \subset R_A$. When A and B are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$, the inclusion $R_A \subset B$ is obtained by theorem 2.■

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