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ERGODIC PROPERTIES OF COMPACT ACTIONS ON C^*–ALGEBRAS

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I. Introduction

Let (A, G, α) be a C^*-dynamical system. In [3] the classical notions of ergodic properties of topological dynamical systems such as topological transitivity, minimality, and uniquely ergodicity are extended and analyzed in the context of non-abelian C^*-dynamical systems. We showed in [2] that if G is a compact group, then minimality, topological transitivity, uniquely ergodicity, and weakly ergodicity of the C^*-dynamical system (A, G, α) are equivalent. But we can give examples which show that the above statement is false if G is not compact. Let M_2^∞ be a Car-algebra, i.e. M_2^∞ is an infinite tensor product $\otimes_{-\infty}^{\infty} M_2$ of full 2×2 matrix algebra M_2. Let α be the shift of M_2^∞ obtained by translating each tensor factor by one to the right. It is clear that $(M_2^\infty, \mathbf{Z}, \alpha)$ is ergodic. Let $p_\infty = \otimes_{-\infty}^{\infty} p$ where p is the non-trivial projection of M_2. Then p_∞ is the α-invariant closed projection in the second dual $M_2^{\infty''}$ of M_2^∞ because the sequence of projections $p_n = \otimes_{-n}^{n} p$ in M_2^∞ is decreasing and converges σ-weakly to p_∞. So $(I - p_\infty)M_2^{\infty''}(I - p_\infty) \cap M_2^\infty$ is the non-zero α-invariant hereditary C^*-subalgebra of M_2^∞ because $I - p_\infty$ is the open projection in $M_2^{\infty''}$. So $(M_2^\infty, \mathbf{Z}, \alpha)$ is not minimal. Let p and q be non-zero orthogonal projections in M_2 with $p + q = I$. Then with these projections we can make two α-invariant hereditary C^*-subalgebras B_1 and B_2 of M_2^∞ with $B_1 B_2 = 0$. This means that $(M_2^\infty, \mathbf{Z}, \alpha)$ is not topologically transitive. Let (A, G, α) and (A, G, β) be C^*-dynamical systems. It is said that two C^*-dynamical sysytems (A, G, α) and (A, G, β) are exterior

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equivalent if there is a function $t \to u_t$ from G to the unitary group of the multiplier algebra $M(A)$ of A satisfying the conditions;

(1) $u_{st} = u_s\alpha_s(u_t)$,
(2) $\beta_t = Ad_{u_t}\alpha_t$,
(3) $t \to u_t x$ is norm continuous for each x in A.

A function satisfying condition (1) is called a unitary cocyle. In W^*-dynamical systems the norm continuity of the condition (3) is replaced with the σ-weak continuity. In this paper we are going to discuss the exterior equivalence of dynamical systems. That is, if two C^*-dynamical systems (or W^*-dynamical systems) are exterior equivalent and one of them has some properties, then what happens to the other ? In [4], it was known that if two C^*-dynamical systems (or W^*-dynamical systems) are exterior equivalent, they have the same Connes' spectrum.

2. Main result

Let (A, G, α) be a C^*-dynamical system. The C^*-dynamical system (A, G, α) is *topologically transitive* if for any non-zero α-invariant hereditary C^*-subalgebras B_1 and B_2 their product B_1B_2 is not zero. If A is the only non-zero α-invariant hereditary C^*-subalgebra of A, then (A, G, α) is called *minimal*, [cf. 3]. A triple (π, u, H) is a covariant representation of (A, G, α) if (π, H) is a representation of A, (u, H) is a unitary representation of G, and

$$\pi(\alpha_t(x)) = u_t\pi(x)u_t^*$$

for all x in A and t in G. For W^*-dynamical systems the useful concepts are of course normal covariant representations.

LEMMA 2.1. *Let two C^*-dynamical systems (A, G, α) and (A, G, β) be exterior equivalent with a unitary cocyle $\{u_g | \, g \in G\}$. If (π, v, H) is a non-degenerate covariant representation of (A, G, α), then $(\pi, \pi(u)v, H)$ is a covariant representation of (A, G, β).*

Proof. Since π is non-degenerate, there is a unique normal homomorphism π'' of the enveloping von Neumann algebra A'' of A onto the

σ-weak closure $\overline{\pi(A)}^{\sigma w}$ of $\pi(A)$ which extends π. Since π is nondegenerate, it is clear that $g \to \pi(u_g)v_g$ is a unitary representation of G. So we have for each $\xi \in H$,

$$\|\pi(u_g)v_g\xi - \xi\| \leq \|\pi(u_g)\|\|v_g\xi - \xi\| + \|\pi(u_g)\xi - \xi\|.$$

Futhermore since $\pi(\beta_g(x)) = \pi(u_g)v_g\pi(x)v_g^*\pi(u_g^*)$ for each $g \in G$, $(\pi, \pi(u)v, H)$ is a covariant representation of (A, G, β).

Let (A, G, α) be a C^*-dynamical system and π be a covariant representation of (A, G, α). Then we can consider the W^*-dynamical system $(\pi(A)'', G, \alpha'')$ on $\pi(A)''$ induced by (A, G, α) where $\alpha''(\pi(x)) = \pi(\alpha(x))$ for all $x \in A$.

PROPOSITION 2.2. *Let C^*-dynamical systems (A, G, α) and (A, G, β) be exterior equivalent with a unitary cocyle $\{u_g \mid g \in G\}$. Let π be a covariant non-degenerate representation of (A, G, α) on a Hilbert space H. Then W^*-dynamical systems $(\pi(A)'', G, \alpha'')$ and $(\pi(A)'', G, \beta'')$ are also exterior equivalent with respect to the sence of W^*-dynamical systems.*

Proof. We only have to show that the function $g \to \pi(u_g)x$ from G into the unitary group of the multiplier algebra $M(\pi(A))$ of $\pi(A)$ is σ-weakly continuous for each x in $\pi(A)''$. For each $x \in \pi(A)''$ we can choose a net $\{x_\alpha\}_{\alpha \in I}$ such that $\{\|x_\alpha\|\}_{\alpha \in I}$ is bounded and $\{x_\alpha\}$ converges σ-strongly to x. Let ω be a positive normal linear functional on $\pi(A)''$. Since $|\omega(y^*x)|^2 \leq \omega(y^*y)\omega(x^*x)$ for all $x, y \in \pi(A)''$, we have

$$|\omega(\pi(u_g)x - x)| \leq \left(\omega(1)\omega((x - x_\alpha)^*(x - x_\alpha))\right)^{\frac{1}{2}} + |\omega(\pi(u_g)x_\alpha - x_\alpha)| + |\omega(x_\alpha - x)|.$$

Since $x_\alpha - x$ converges σ-strongly to 0, $(x_\alpha - x)^*(x_\alpha - x)$ converges σ- weakly to 0. Hence we see that $\pi(u_g)x$ converges σ-weakly to x for each $x \in \pi(A)''$ as g goes to the identity of G.

LEMMA 2.3. *Let (A, G, α) be a C^*-dynamical system. Let p and q be equivalent projections in the fixed point algebra $M(A)^\alpha$ of the multiplier algebra $M(A)$. Assume that $(qAq, G, \alpha|_{qAq})$ is topologically transitive. Then $(pAp, G, \alpha|_{pAp})$ is also topologically transitive.*

Proof. Suppose that $(pAp, G, \alpha|_{pAp})$ is not topologically transitive. Then there exist non-zero two elements x and y in pAp such that

$$x\alpha_g(y) = 0, \qquad g \in G.$$

Since p and q are equivalent in $M(A)^\alpha$, there exists a partial isometry v in $M(A)^\alpha$ such that

$$v^*v = p, \qquad vv^* = q.$$

Since x and y are contained in pAp, we have $x\alpha_g(y) = pxp\alpha_g(pyp)$. By α-invariance of p

$$x\alpha_g(y) = v^*vxv^*v\alpha_g(y)v^*v = 0$$

for all $g \in G$. Since v is fixed by α_g for all $g \in G$, we get for all $g \in G$

$$\begin{aligned} 0 =& v(v^*vxv^*v\alpha_g(y)v^*v)v^* \\ =& qvxv^*q\alpha_g(qvyv^*q). \end{aligned}$$

Put $x' = qvxv^*q$ and $y' = qvyv^*q$. Then x' and y' are non-zero elements in qAq. From the above calculation, $x'\alpha_g(y') = 0$ for all $g \in G$. Therefore $(qAq, G, \alpha|_{qAq})$ is not topologically transitive.

Let (A, G, α) be a C^*-dynamical system. When G is a compact group, A can be represented faithfully and covariantly.

LEMMA 2.4. *Let (A, G, α) be a C^*-dynamical system and G be a compact group. Let A be represented faithfully and covariantly. If (A, G, α) is topologically transitive, then the W^*-dynamical system (A'', G, α'') induced by (A, G, α) is ergodic and the von Neumann algebra A'' is finite.*

Proof. Let (A'', G, α'') be the W^*-dynamical system induced by the C^*-dynamical system (A, G, α). Let P_0 be the conditional expectation from A'' to the fixed point algebra $A''^{\alpha''}$ defined by

$$P_0(x) = \int_G \alpha''_g(x)dg$$

for all $x \in A''$. Since P_0 is σ-weak continuous and A is σ-weak dense in A'', $P_0(A) = A^\alpha$ is σ-weak dense in $A''^{\alpha''}$. By Corollary 2.3 in [2] A^α is of one dimension, so $A''^{\alpha''}$ is also trivial. Therefore (A'', G, α'') is ergodic and by Corollary 4.2 in [1] A'' is finite.

Let M be a finite von Neumann algebra and $Z(M)$ be its center. The map $\tau : M \to Z(M)$ is called a *canonical central trace* if it satisfies the following conditions:

(1) τ is linear and bounded,
(2) $\tau(xy) = \tau(yx)$ for any $x, y \in M$,
(3) $\tau(z) = z$ for any $z \in Z(M)$.

Let M be a finite von Neumann algebra and τ be the canonical central trace on M. Let p and q be projections in M. $\tau(p) = \tau(q)$ if and only if p and q are equivalent [cf 6].

LEMMA 2.5. *Let (M, G, α) be a W^*-dynamical system and G be a compact group. Let A be the set defined by*

$$A = \{x \in M \mid x \to \alpha_g(x) \text{ is norm continuous}\}.$$

If (M, G, α) is ergodic, the C^*-dynamical system $(A, G, \alpha|_A)$ is topologically transitive.

Proof. It is known that A is the α-invariant C^*-algebra. Since G is compact, there exists a σ-weakly continuous expectation P_0 from M to the fixed point algebra M^α as in the proof of Lemma 2.4. Since A is σ-weak dense in M, $P_0(A)$ is σ-weak dense in M^α. Therefore A^α is trivial. By Corollary 2.3 in [2] $(A, G, \alpha|_A)$ is topologically transitive.

THEOREM 2.6. *Let W^*-dynamical systems (M, G, α) and (M, G, β) be exterior equivalent and G be a compact group. Let (M, G, α) be ergodic. If $Z(M^\beta) = Z(M)^\beta$, then the W^*-dynamical system (M, G, β) is also ergodic.*

Proof. Put $p = I \otimes e_{11}$ and $q = I \otimes e_{22}$ where $\{e_{ij} \mid i, j = 1, 2\}$ is the matrix unit of M_2. Since (M, G, α) and (M, G, β) are exterior equivalent, there exists the W^*-dynamical system $(M \otimes M_2, G, \gamma)$ such that

$$\gamma_g \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \alpha_g(x) & 0 \\ 0 & \beta_g(y) \end{pmatrix}$$

for $x, y \in M$ and $g \in G$. It is not difficult to show that

$$Z((M \otimes M_2)^{\gamma}) \subset \begin{pmatrix} Z(M^{\alpha}) & o \\ 0 & Z(M^{\beta}) \end{pmatrix}.$$

Since $Z(M^{\beta}) = Z(M)^{\beta}$, we have

$$Z(M \otimes M_2)^{\gamma} = Z((M \otimes M_2)^{\gamma}).$$

Since $M \otimes M_2$ is finite, we consider the canonical central trace $\tau : M \otimes M_2 \rightarrow Z(M \otimes M_2)$. We consider the restriction map

$$\tau|_{(M \otimes M_2)^{\gamma}} : (M \otimes M_2)^{\gamma} \rightarrow Z(M \otimes M_2)^{\gamma},$$

and denote it by τ^{γ}. Since $Z(M \otimes M_2)^{\gamma} = Z((M \otimes M_2)^{\gamma})$, τ^{γ} becomes the canonical central trace on $(M \otimes M_2)^{\gamma}$. Since p and q are equivalent in $M \otimes M_2$, we have $\tau(p) = \tau(q)$. Since p and q are contained in $(M \otimes M_2)^{\gamma}$, we have $\tau^{\gamma}(p) = \tau^{\gamma}(q)$. Hence p and q are equivalent in $(M \otimes M_2)^{\gamma}$. So we can choose a partial isometry v in $(M \otimes M_2)^{\gamma}$ such that $v^*v = p$ and $vv^* = q$. Let A be the set defined as follows :

$$A = \{x \in M |\ x \rightarrow \alpha_g(x) \text{ is norm continuous}\}.$$

Then A is the α-invariant C^*-algebra and σ-weak dense in M and $A \otimes M_2$ is σ-weak dense in $M \otimes M_2$. Since $(p(A \otimes M_2)p, G, \gamma|_{p(A \otimes M_2)p})$ is isomorphic to (A, G, α), $(p(A \otimes M_2)p, G, \gamma|_{p(A \otimes M_2)p})$ is topologically transitive by Lemma 2.5. Since p and q are equivalent with γ-invariant partial isometry v, $(q(M \otimes M_2)q, G, \gamma|_{q(M \otimes M_2)q})$ is also topologically transitive by Lemma 2.3. Hence by Lemma 2.4 (M, G, β) is ergodic.

COROLLARY 2.7. *Let a W^*-dynamical system (M, G, α) be ergodic and G be a compact abelian group. Let (M, G, α) be exterior equivalent to a W^*-dynamical system (M, G, β). If $Z(M^{\beta}) = Z(M)^{\beta}$, then* $\mathrm{Sp}(\alpha) = \mathrm{Sp}(\beta)$.

Proof. Since (M, G, α) and (M, G, β) are exterior equivalent, it was known in [4] that $\Gamma(\alpha) = \Gamma(\beta)$. By Theorem 2.6, we have $\mathrm{Sp}(\alpha) = \mathrm{Sp}(\beta)$.

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