Bull. Korean Math. Soc. **31** (1994), No. 2, pp. 269–288

ON NEW CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f * g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

Ruscheweyh [12] introduced the classes K_n of functions $f(z) \in A$ satisfying

$$\text{Re}\left\{\frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}}\right\} > \frac{n+1}{2} \tag{1.4}$$

for $n \in N_0 = NU\{0\}$ and $z \in U$, where $N = \{1, 2, \cdots\}$. Ruscheweyh [12] showed the basic property

$$k_{n+1} \subset k_n, \quad n \in N_0. \tag{1.5}$$

Received June 3, 1993.

Key words-Analytic, Ruscheweyh derivative, fractional calculus.

AMS (1991) Subject Classification. 30C45.

Note that K_0 is the class $S^*\left(\frac{1}{2}\right)$ of starlike functions of order $\frac{1}{2}$.

Let

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \; n \in N_0. \tag{1.6}$$

This symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. Using Hadamard product, Ruscheweyh [12] observed that if

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z) \quad (\alpha \geq -1) \tag{1.7}$$

then (1.6) is equivalent to (1.7) when $\alpha = n \in N_0$. Thus it follows from (1.4) that the necessary and sufficient condition for $f(z) \in A$ to belong to K_n is

$$\text{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \frac{1}{2}, \qquad (z \in U). \tag{1.8}$$

Note that K_{-1} is the class of functions $f(z) \in A$ satisfying

$$\text{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2}, \qquad (z \in U). \tag{1.9}$$

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.10}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.11}$$

Let T denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.12}$$

In [10] Owa studied the classes $K_n^*(n \in N_0)$ by using the n-th order Ruscheweyh derivative of $f(z)$, defined as follows:

DEFINITION 1. We say that the function $f(z)$ defined by (1.12) is in the class K_n^* if $f(z)$ statisfies the condition (1.8) for $n \in N_0$.

In order to show our results, we need the following lemmas given by Owa [10].

LEMMA 1. *Let the function $f(z)$ defined by (1.12). Then $f(z) \in K_n^*$ if and only if*

$$\sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1} \right) \delta(n,k) a_k \leq 1 \tag{1.13}$$

for $n \in N_0$. The result is sharp for the function

$$f(z) = z - \frac{(n+1)}{(2k+n-1)\delta(n,k)} z^k \quad (k \geq 2). \tag{1.14}$$

LEMMA 2. *The extreme points of the class K_n^* are $f_1(z) = z$ and $f_k(z) = z - \frac{(n+1)}{(2k+n-1)\delta(n,k)} z^k \quad (k \geq 2)$.*

2. Some properties of the class K_n^*

THEOREM 1. *$K_{n+1}^* \subset K_n^*$ for each $n \in N_0$.*

Proof. Let the function $f(z)$ defined by (1.12) be in the class K_{n+1}^* ; then

$$\sum_{k=2}^{\infty} \left(\frac{2k+n}{n+2} \right) \delta(n+1,k) a_k \leq 1 \tag{2.1}$$

and since

$$\delta(n,k) \leq \delta(n+1,k) \quad \text{for} \quad k \geq 2, \tag{2.2}$$

we have

$$\sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1} \right) \delta(n,k) a_k \leq \sum_{k=2}^{\infty} \left(\frac{2k+n}{n+2} \right) \delta(n+1,k) a_k \leq 1 \tag{2.3}$$

The result follows from Lemma 1.

THEOREM 2. *The class K_n^* is closed under convex linear combinations.*

Proof. Let the functions

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0,\ i = 1, 2) \tag{2.4}$$

be in the class K_n^*. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \tag{2.5}$$

is in the class K_n^*. Since

$$h(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} z^k, \tag{2.6}$$

with the aid of Lemma 1, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} \\ =&\lambda \sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) a_{k,1} \\ &+ (1-\lambda) \sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) a_{k,2} \leq 1 \end{aligned}$$

which implies that $h(z) \in K_n^*$.

3. Integral operators

THEOREM 3. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \tag{3.1}$$

also belongs to the class K_n^*.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{3.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{3.3}$$

Therefore,

$$\begin{aligned}\sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) b_k &= \sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) \left(\frac{c+1}{c+k}\right) a_k \\ &\leq \sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) a_k \leq 1,\end{aligned} \tag{3.4}$$

since $f(z) \in K_n^*$. Hence, by Lemma 1, $F(z) \in K_n^*$.

THEOREM 4. *Let the function $F(z)$ defined by (1.12) be in the class K_n^*, and let c be a real number such that $c > -1$. Then the function $f(z)$ defined by (3.1) is univalent in $|z| < r^*$, where*

$$r^* = \inf_k \left[\frac{(c+1)(2k+n-1)\delta(n,k)}{k(c+k)(n+1)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{3.5}$$

The result is sharp.

Proof. From (3.1), we have

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{(c+1)} \quad (c > -1) \tag{3.6}$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k. \tag{3.7}$$

In order to obtain the required result it suffices to show that

$$|f'(z)-1|<1 \text{ in } |z|<r^*.$$

Now

$$|f'(z)-1| \le \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} < 1. \tag{3.8}$$

But Lemma 1 confirms that

$$\sum_{k=2}^{\infty} \left(\frac{2k+n-1}{n+1}\right) \delta(n,k) a_k \le 1. \tag{3.9}$$

Hence (3.8) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{(c+1)} < \frac{(2k+n-1)\delta(n,k)}{(n+1)}$$

or if

$$|z| < \left[\frac{(c+1)(2k+n-1)\delta(n,k)}{k(c+k)(n+1)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{3.10}$$

Therefore $f(z)$ is univalent in $|z|<r^*$. Sharpness follows if we take

$$f(z) = z - \frac{(n+1)(c+k)}{(2k+n-1)\delta(n,k)(c+1)} z^k \quad (k \ge 2). \tag{3.11}$$

4. Radii of close-to-convexity, starlikeness and convexity

THEOREM 5. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*, then $f(z)$ is close-to-convex of order $\rho (0 \le \rho < 1)$ in $|z| < r_1(n, \rho)$, where*

$$r_1(n,\rho) = \inf_k \left[\frac{(1-\rho)(2k+n-1)\delta(n,k)}{k(n+1)} \right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{4.1}$$

The result is sharp, with the extremal function $f(z)$ given by (1.14).

Proof. We must show that $|f'(z) - 1| \le 1 - \rho$ for $|z| < r_1(n, \rho)$. We have

$$|f'(z) - 1| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \le 1 - \rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \le 1. \tag{4.2}$$

Hence, by (3.9), (4.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\rho)} \le \frac{(2k+n-1)\delta(n,k)}{(n+1)}$$

or if

$$|z| \le \left[\frac{(1-\rho)(2k+n-1)\delta(n,k)}{k(n+1)} \right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{4.3}$$

The theorem follows easily from (4.3).

THEOREM 6. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*, then $f(z)$ is starlike of order $\rho (0 \le \rho < 1)$ in $|z| < r_2(n, \rho)$, where*

$$r_2(n,\rho) = \inf_k \left[\frac{(1-\rho)(2k+n-1)\delta(n,k)}{(k-\rho)(n+1)} \right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{4.4}$$

The result is sharp, with the extremal function $f(z)$ given by (1.14).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ for $|z| < r_2(n, \rho)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{(1-\rho)} \leq 1. \tag{4.5}$$

Hence, by (3.9), (4.5) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{(1-\rho)} \leq \frac{(2k+n-1)\delta(n,k)}{(n+1)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(2k+n-1)\delta(n,k)}{(k-\rho)(n+1)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.6}$$

The theorem follows easily from (4.6).

COROLLARY 1. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*, then $f(z)$ is convex of order $\rho (0 \leq \rho < 1)$ in $|z| < r_3(n, \rho)$, where*

$$r_3(n,\rho) = \inf_k \left[\frac{(1-\rho)(2k+n-1)\delta(n,k)}{k(k-\rho)(n+1)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.7}$$

The result is sharp, with the extremal function $f(z)$ given by (1.14).

5. Fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (c.f., e.g., [2], [5, Chapter 13], [6], [7], [8], [11], [13], [14], [15, p. 28 et seq.], [17], [18], [19] and [21]). We find it to be convenient to recall here the following definitions which were used recently by Owa [9] (and by Srivastava and Owa [16]).

DEFINITION 2. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad (\lambda > 0), \tag{5.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

DEFINITION 3. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \quad (0 \leq \lambda < 1), \tag{5.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed, as in Definition 2.

DEFINITION 4. Under the hypotheses of Definition 3, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1; n \in N_0). \tag{5.3}$$

THEOREM 7. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*. Then we have*

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{2}{(n+3)(2+\lambda)}|z|\right\} \tag{5.4}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{2}{(n+3)(2+\lambda)}|z|\right\} \tag{5.5}$$

for $\lambda > 0$, $n \in N_0$, and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) =&\Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z) \\ =&z-\sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} a_k z^k = z-\sum_{k=2}^{\infty} \Psi(k) a_k z^k, \end{aligned} \tag{5.6}$$

where

$$\Psi(k)=\frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} \quad (k \geq 2). \tag{5.7}$$

Since

$$0<\Psi(k) \leq \Psi(2)=\frac{2}{2+\lambda}, \tag{5.8}$$

by using Lemma 1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{1}{n+3}. \tag{5.9}$$

Therefore, by using (5.8) and (5.9), we can see that

$$\left|F(z)\right| \geq |z|-\psi(2)|z|^2 \sum_{k=2}^{\infty} a_k \geq |z|-\frac{2}{(n+3)(2+\lambda)}|z|^2 \tag{5.10}$$

and

$$\left|F(z)\right| \leq |z|+\psi(2)|z|^2 \sum_{k=2}^{\infty} a_k \leq |z|+\frac{2}{(n+3)(2+\lambda)}|z|^2 \tag{5.11}$$

which prove the inequalities of theorem 7. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{-\lambda} f(z)=\frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{2}{(n+3)(2+\lambda)} z\right\} \tag{5.12}$$

or

$$f(z)=z-\frac{1}{n+3} z^2. \tag{5.13}$$

COROLLARY 2. *Under the hypotheses of Theorem 7, $D_z^{-\lambda} f(z)$ is included in the disc with center at the origin and radius R_1 given by*

$$R_1 = \frac{1}{\Gamma(2+\lambda)} \left\{ 1 + \frac{2}{(n+3)(2+\lambda)} \right\}. \tag{5.14}$$

THEOREM 8. *Let the function $f(z)$ defined by (1.12) be in the class K_n^*. Then we have*

$$\left| D_z^{\lambda} f(z) \right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{2}{(n+3)(2-\lambda)} |z| \right\} \tag{5.15}$$

and

$$\left| D_z^{\lambda} f(z) \right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 + \frac{2}{(n+3)(2-\lambda)} |z| \right\} \tag{5.16}$$

for $0 \leq \lambda < 1$, $n \in N_0$, and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} G(z) =& \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \qquad (5.17) \\ =& z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k \, z^k = z - \sum_{k=2}^{\infty} \phi(k) k \, a_k \, z^k, \end{aligned}$$

where

$$\phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq 2). \tag{5.18}$$

Since

$$0 < \phi(k) \leq \phi(2) = \frac{1}{2-\lambda}, \tag{5.19}$$

by using Lemma 1, we have

$$\sum_{k=2}^{\infty} k \, a_k \leq \frac{2}{n+3}. \tag{5.20}$$

Therefore, by using (5.19) and (5.20), we can see that

$$|G(z)| \geq |z| - \phi(2)|z|^2 \sum_{k=2}^{\infty} k\ a_k \geq |z| - \frac{2}{(n+3)(2-\lambda)}|z|^2 \qquad (5.21)$$

and

$$|G(z)| \leq |z| + \phi(2)|z|^2 \sum_{k=2}^{\infty} k\ a_k \leq |z| + \frac{2}{(n+3)(2-\lambda)}|z|^2 \qquad (5.22)$$

which give the inequalities of Theorem 8. Since equalities are attained for the function $f(z)$ defined by

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{2}{(n+3)(2-\lambda)} z\right\} \qquad (5.23)$$

that is, by (5.13), we complete the assertion of Theorem 8.

COROLLARY 3. *Under the conditions of Theorem 8, $D_z^{\lambda} f(z)$ is included in the disc with center at the origin and radius R_2 given by*

$$R_2 = \frac{1}{\Gamma(2-\lambda)} \left\{1 + \frac{2}{(n+3)(2-\lambda)}\right\}. \qquad (5.24)$$

6. Fractional integral operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [20].

DEFINITION 5. For real numbers $\beta > 0$, γ and η, the fractional integral operator $I_{0,z}^{\beta,\gamma,\eta}$ is defined by

$$I_{0,z}^{\beta,\gamma,\eta} f(z) = \frac{z^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} F\Big(\beta+\gamma, -\eta; \beta; 1-\frac{t}{z}\Big) f(t)dt \quad (6.1)$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = 0\big(|z|^{\varepsilon}\big),\ z \to 0,$$

where

$$\varepsilon > \text{ Max } (0, \gamma - \eta) - 1,$$
$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{6.2}$$

where $(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0) \\ \nu(\nu + 1) \cdots (\nu + k - 1) & (k \in N), \end{cases} \tag{6.3}$$

and the multiplicity of $(z - t)^{\beta - 1}$ is removed by requiring $\log(z - t)$ to be real when $z - t > 0$.

REMARK. For $\gamma = -\beta$, we note that

$$I_{0,z}^{\beta, -\beta, \eta} f(z) = D_z^{-\beta} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [20].

LEMMA 3. *If $\beta > 0$ and $k > \gamma - \eta - 1$, then*

$$I_{0,z}^{\beta, \gamma, \eta} z^k = \frac{\Gamma(k + 1)\Gamma(k - \gamma + \eta + 1)}{\Gamma(k - \gamma + 1)\Gamma(k + \beta + \eta + 1)} z^{k - \gamma}. \tag{6.4}$$

With the aid of Lemma 3, we prove

THEOREM 9. *Let $\beta > 0, \gamma < 2, \beta + \eta > -2, \gamma - \eta < 2, \gamma(\beta + \eta) \leq 3\beta$. If the function $f(z)$ defined by (1.12) is in the class K_n^*, then*

$$\left| I_{0,z}^{\beta, \gamma, \eta} f(z) \right| \tag{6.5}$$
$$\geq \frac{\Gamma(2 - \gamma + \eta) |z|^{1 - \gamma}}{\Gamma(2 - \gamma)\Gamma(2 + \beta + \eta)} \left\{ 1 - \frac{2(2 - \gamma + \eta)}{(n + 3)(2 - \gamma)(2 + \beta + \eta)} |z| \right\}$$

and

$$\left|I_{0,z}^{\beta,\gamma,\eta} f(z)\right| \tag{6.6}$$
$$\leq \frac{\Gamma(2-\gamma+\eta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} \left\{1 + \frac{2(2-\gamma+\eta)}{(n+3)(2-\gamma)(2+\beta+\eta)}|z|\right\}$$

for $z \in U_0$, *where*

$$U_0 = \begin{cases} U & (\gamma \leq 1) \\ U - \{0\} & (\gamma > 1). \end{cases}$$

The equalities in (6.5) and (6.6) are attained by the function $f(z)$ *given by (5.13).*

Proof. By using Lemma 3, we have

$$I_{0,z}^{\beta,\gamma,\eta} f(z) = \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} z^{1-\gamma} - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k \, z^{k-\gamma}. \tag{6.7}$$

Letting

$$H(z) = \frac{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} \, I_{0,z}^{\beta,\gamma,\eta} f(z) = z - \sum_{k=2}^{\infty} h(k) a_k \, z^k, \tag{6.8}$$

where

$$h(k) = \frac{(2-\gamma+\eta)_{k-1}(1)_k}{(2-\gamma)_{k-1}(2+\beta+\eta)_{k-1}} \quad (k \geq 2), \tag{6.9}$$

we can see that $h(k)$ is non-increasing for integers $k \geq 2$, and we have

$$0 < h(k) \leq h(2) = \frac{2(2-\gamma+\eta)}{(2-\gamma)(2+\beta+\eta)}. \tag{6.10}$$

Therefore, by using (5.9) and (6.10), we have

$$
\begin{aligned}
|H(z)| \geq & |z| - h(2)|z|^2 \sum_{k=2}^{\infty} a_k \\
\geq & |z| - \frac{2(2-\gamma+\eta)}{(n+3)(2-\gamma)(2+\beta+\eta)}|z|^2
\end{aligned}
\tag{6.11}
$$

and

$$
\begin{aligned}
|H(z)| \leq & |z| + h(2)|z|^2 \sum_{k=2}^{\infty} a_k \\
\leq & |z| + \frac{2(2-\gamma+\eta)}{(n+3)(2-\gamma)(2+\beta+\eta)}|z|^2.
\end{aligned}
\tag{6.12}
$$

This completes the proof of Theorem 9.

REMARK. Taking $\beta = -\gamma = \lambda$ in Theorem 9, we get the result of Theorem 7.

7. Modified Hadamard Product

Let the functions $f_i(z)$ $(i = 1, 2)$ be defined by (2.4). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$
f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1}\ a_{k,2} z^k. \tag{7.1}
$$

THEOREM 10. *Let $f_1(z) \in K^*_{n_1}$ and $f_2(z) \in K^*_{n_2}$. Then the modified Hadamard product $f_1 * f_2(z) \in K^*_{n_1} \cap K^*_{n_2}$.*

Proof. Since $f_2(z) \in K^*_{n_2}$, we have from (5.9) that

$$
a_{k,2} \leq \frac{1}{n_2 + 3}. \tag{7.2}
$$

From Lemma 1, since $f_1(z) \in K^*_{n_1}$, we have

$$
\sum_{k=2}^{\infty} \left(\frac{2k + n_1 - 1}{n_1 + 1} \right) \delta(n_1, k) a_{k,1} \leq 1. \tag{7.3}
$$

Now, from (7.2) and (7.3),

$$\sum_{k=2}^{\infty}\left(\frac{2k+n_1-1}{n_1+1}\right)\delta(n_1,k)a_{k,1}\ a_{k,2}$$

$$\leq\frac{1}{n_2+3}\sum_{k=2}^{\infty}\left(\frac{2k+n_1-1}{n_1+1}\right)\delta(n_1,k)a_{k,1}\leq\frac{1}{n_2+3}\leq 1.$$

Hence $f_1 * f_2(z) \in K^*_{n_1}$. Interchanging n_1 and n_2 by each other in the above, we get $f_1 * f_2(z) \in K^*_{n_2}$. Hence the theorem.

8. Linear combination of functions in the class K^*_n

THEOREM 11. *Let each of the functions $f_1(z), f_2(z), \cdots, f_m(z)$ defined by*

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i}\ z^k (a_{k,i} \geq 0;\ i = 1, 2, \cdots, m) \tag{8.1}$$

be in the same class K^*_n. Then the function $h(z)$ given by

$$h(z) = \frac{1}{m}\sum_{i=1}^{m} f_i(z) \tag{8.2}$$

is also in the class K^*_n.

Proof. By the definition (8.2) of $h(z)$, we have the expansion

$$h(z) = z - \sum_{k=2}^{\infty}\left[\frac{1}{m}\sum_{i=1}^{m} a_{k,i}\right] z^k. \tag{8.3}$$

Since $f_i(z) \in K^*_n$ for every $i = 1, 2, \cdots, m$, by using Lemma 1, we obtain

$$\sum_{k=2}^{\infty}\left(\frac{2k+n-1}{n+1}\right)\delta(n,k)\left[\frac{1}{m}\sum_{i=1}^{m} a_{k,i}\right] \leq 1, \tag{8.4}$$

which, in view of Lemma 1, yields Theorem 11.

9. Support points

A function $f(z)$ in K_n^* is said to be a support point of K_n^* if there exists a continuous linear functional J on T such that Re $(J(f)) \geq$ Re $(J(g))$ for all $g(z) \in K_n^*$, and Re (J) is non-constant on K_n^*. We denote by Supp K_n^* the set of support points of K_n^*, and by Ext K_n^* the set of extreme points of K_n^*.

Let F be a subfamily of univalent functions in the unit disc U whose set of extreme points is countable, suppose $f_0(z)$ is a support point of F, and let J be a corresponding continuous linear functional. Let

$$G_J = \{f \in F : \text{Re}\,(J(f)) = \text{Re}\,(J(f_0))\}. \tag{9.1}$$

Then Deeb [4] showed the following lemma.

LEMMA 4. *Let G_J be defined by (9.1). Then G_J is convex, Ext $G_J \subset$ Ext F, and*

$$G_J = \left\{ f \in F : f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),\ \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1, f_k(z) \in \text{ Ext } G_J \right\}. \tag{9.2}$$

Let A_1 be the class of functions of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the unit disc U. Then, Brickman, MacGregor and Wilken [3] have proved the following result.

LEMMA 5. *Let $\{b_k\}$ be a sequence of complex numbers such that*

$$\lim_{k\to\infty} \sup |b_k|^{\frac{1}{k}} < 1,$$

and set $J(f) = \sum_{k=0}^{\infty} a_k b_k$ for $f(z) \in A_1$. Then J is a continuous linear functional on A_1.

Conversely, any continuous linear functional on A_1 is given by such a sequence $\{b_k\}$.

Now, we prove

THEOREM 12. *The set Supp K_n^* of support points of K_n^* is given by*

$$Supp\ K_n^* = \Big\{ f \in K_n^* : f(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k,$$

$$\lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k \leq 1,\ \lambda_j = 0 \text{ for some } j \Big\}.$$

Proof. Let

$$f_0(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k \tag{9.3}$$

be in the class K_n^*, where $\lambda_k \geq 0$, $\sum_{k=2}^{\infty} \lambda_k \leq 1$, and $\lambda_j = 0$ for some $j \geq 2$. If $b_k = 0$ $(k \geq 2,\ k \neq j)$ and $b_1 = b_j = 1$, then

$$\lim_{k \to \infty} \sup |b_k|^{\frac{1}{k}} < 1.$$

Therefore, by using Lemma 5, we define the continuous linear functional J given by $\{b_k\}$. It follows that $J(f_0) = 1$ and $J(f) = 1 - a_j \leq 1$ for $f(z) \in K_n^*$. This implies that Re $(J(f_0)) \geq$ Re $(J(f))$ for all $f(z) \in K_n^*$. Hence, $f_0(z)$ is a support point of the class K_n^*.

Conversely, suppose that $f_0(z)$ is a support point of K_n^*, and that its continuous linear functional J is given by $\{b_k\}$. Then Re (J) is also continuous and linear on K_n^*. Therefore, by the Krein - Milman theorem, there exists an extreme point $f_k(z)$ of K_n^* such that

$$\text{Re } (J(f_0)) = \text{Max } \{\text{Re } (J(f)) : f \in K_n^*\} = \text{Re } (J(f_k)). \tag{9.4}$$

Let

$$G_J = \{f_k : \text{Re } (J(f_0)) = \text{Re } (J(f_k)), f_k \in \text{ Ext } K_n^*\}.$$

Then, we note that Ext K_n^* is countable by means of Lemma 2. If $G_J =$ Ext K_n^*, then Re (J) must be constant on K_n^*. This contradicts that $f_0(z)$ is a support point of K_n^*. Consequently, there exists a j such that Re $(J(f_0)) >$ Re $(J(f_j))$. It follows from this fact that

$$\text{Ext } G_J \subset \{f_k : f_k \in \text{Ext } K_n^*,\ k \geq 2, k \neq j\}. \tag{9.5}$$

Consequently, with the help of Lemma 4, we have

$$f_0(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \tag{9.6}$$

where $\lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k = 1$, and $f_k(z) \in \text{Ext } G_J,\ k \geq 2, k \neq j$.

Further, by using Lemma 2, we obtain

$$f_0(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k \tag{9.7}$$

which completes the proof of Theorem 12.

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