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ON THE ME–CONNECTION IN MEX_n

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I. Introduction

The usual Einstein's n-dimensional unified field theory is based on a generalized n-dimensional Riemannian manifold X_n, which is referred to a real coordinate system x^ν and obeys coordinate transformations $x^\nu \longrightarrow \bar{x}^\nu$ (*) for which $Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0$.

The algebraic structure on X_n is endowed with a general real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{1.1}$$

where

$$\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0. \tag{1.2}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{1.3}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner.

The differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a real general connection $\Gamma^\nu_{\lambda\mu}$, which satisfies the transformation rule

$$\bar{\Gamma}^\nu_{\lambda\mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\right) \tag{1.4}$$

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(*) Throughout the present paper, Greek indices are used for the holomonic components of tensors in X_n. They take the values $1, 2, \cdots, n$ unless stated otherwise, and follow the summation convention.

and the system of Einstein's equations

$$\partial_\omega g_{\lambda\mu} - \Gamma^\alpha_{\lambda\omega} g_{\alpha\mu} - \Gamma^\alpha_{\omega\mu} g_{\lambda\alpha} = 0, \tag{1.5a}$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}, \tag{1.5b}$$

where D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^\nu = \Gamma^\nu_{[\lambda\mu]} \tag{1.6}$$

is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$.

The Einstein's connection $\Gamma^\nu_{\lambda\mu}$ which takes the form

$$\Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2\delta_\lambda{}^\nu X_\mu - 2g_{\lambda\mu} X^\nu \tag{1.7}$$

for a non-null vector X^ν is called *ME-connection*, and a generalized Riemannian manifold X_n connected by this connection is called *n-dimensioal ME-manifold*. We denote it by MEX_n. In the representation of ME-connection the vector X^ν will be called *ME-vector*.

The following quantities will be frequently used in our subsequent considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}} \quad , \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{1.8}$$

$$\sigma = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases} \tag{1.9}$$

$${}^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\lambda{}^\nu, \quad (p = 1, 2, 3, \cdots), \tag{1.10}$$

$$K_0 = 1, \quad K_p = k^{\alpha_1}_{[\alpha_1} k^{\alpha_2}_{\alpha_2} \cdots k^{\alpha_p}_{\alpha_p]}, \quad (p = 0, 1, 2, \cdots), \tag{1.11}$$

$$K_{\omega\mu\nu} = \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu}, \tag{1.12}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$ defined by $h_{\lambda\mu}$.

It has been shown that the following relations hold in X_n (see [5]):

$$K_p = 0 \quad \text{if } p \text{ is odd}, \quad K_n = k \quad \text{if } n \text{ is even}, \tag{1.13}$$

$$Det\,(Mh_{\lambda\mu} + k_{\lambda\mu}) = \mathfrak{h} \sum_{s=0}^{n-\sigma} K_s M^{n-s}, \quad (M \text{ is a real number}), \tag{1.14}$$

$$\sum_{n=0}^{n-\sigma} K_s {}^{(n+p-s)}k_\lambda{}^\nu = 0, \quad (p = 0, 1, 2, \cdots), \tag{1.15}$$

(Generalized recurrence relation in X_n).

Here and in what follows, *the induces s and t are assumed to take the values $0, 2, 4, 6, \cdots$ in the specified range.*

If the system (1.5) admits a solution $\Gamma^\nu_{\lambda\mu}$, it must be of the form

$$\Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu}, \tag{1.16}$$

where

$$U^\nu{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}. \tag{1.17}$$

The skew-symmetric part of (1.5) gives the following relation satisfied by the torsion tensor $S_{\omega\mu\nu}$:

$$B_{\omega\mu\nu} = S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{112}{S}{}_{\omega\mu\nu}, \tag{1.18}$$

where

$$B_{\omega\mu\nu} = \frac{1}{2}\left(K_{\omega\mu\nu} + 3K_{[\alpha\beta\gamma]} k_\omega{}^\alpha k_\mu{}^\beta k_\nu{}^\gamma\right), \tag{1.19}$$

$$\overset{pqr}{S}{}_{\omega\mu\nu} = S_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha\,(q)}k_\mu{}^{\beta\,(r)}k_\nu{}^\gamma, \quad (p, q, r = 0, 1, 2, \cdots). \tag{1.20}$$

Furthermore, the tensor field $k_{\lambda\mu}$ is said to belong to

(1) the first class if $K_{n-\sigma} \neq 0$,
(2) the second class with j^{th} category if $K_{2j} \neq 0$, $K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0$,
(3) the third class if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0$.

II. The ME-connection $\Gamma^{\nu}_{\lambda\mu}$.

This section is denoted mainly to the proof of a necessary and sufficient condition for the existence of ME-connection and to that of an useful representation of the ME-connection.

AGREEMENT 2.1. (a) All considerations in the present section are dealt for a general $n \geq 2$ and for all possible classes.

(b) In our further considerations, we use the word " present condition " to describe the situations that Einstein's connection, given by (1.16) and (1.17), takes the form (1.7).

THEOREM 2.2. *A necessary and sufficient condition that the present condition holds is that for a non-null vector* X^{ν}

(a) the torsion tensor $S_{\lambda\mu}{}^{\nu}$ *is given by*

$$S_{\lambda\mu}{}^{\nu} = 2\delta_{[\lambda}{}^{\nu}X_{\mu]} - 2k_{\lambda\mu}X^{\nu}, \tag{2.1}$$

(b) and the tensor field $g_{\lambda\mu}$ *satisfies*

$$\delta_{(\lambda}{}^{\nu}X_{\mu)} - h_{\lambda\mu}X^{\nu} = k_{(\lambda}{}^{\nu}X_{\mu)} + 2k_{(\lambda}{}^{\nu}k_{\mu)}{}^{\alpha}X_{\alpha}, \tag{2.2a}$$

or equivalently

$$g_{\nu(\lambda}X_{\mu)} + 2k_{\nu(\lambda}k_{\mu)}{}^{\alpha} - h_{\lambda\mu}X_{\nu} = 0. \tag{2.2b}$$

Proof. Suppose that the present condition holds. Then, in virtue of (1.16) and (1.7), we have

$$S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} = \delta_{\lambda}{}^{\nu}X_{\mu} - 2g_{\lambda\mu}X^{\nu},$$

or equivalently

$$S_{\lambda\mu}{}^{\nu} = 2\delta_{[\lambda}{}^{\nu}X_{\mu]} - 2k_{\lambda\mu}X^{\nu}, \tag{2.3a}$$

$$U^{\nu}{}_{\lambda\mu} = 2\delta_{(\mu}{}^{\nu}X_{\mu)} - 2h_{\lambda\mu}X^{\nu}. \tag{2.3b}$$

On the other hand, (1.17) and (2.3a) give

$$U^{\nu}{}_{\lambda\mu} = 2k_{(\mu}{}^{\nu}X_{\lambda)} + 4k_{(\lambda}{}^{\nu}k_{\mu)}{}^{\alpha}X_{\alpha}. \tag{2.4}$$

The condition (2.2a) immediately follows from (2.3b) and (2.4).

Conversely, suppose that the statements (a) and (b) hold. Then

$$\begin{aligned}
U^{\nu}{}_{\lambda\mu} &= 2\delta_{(\lambda}{}^{\nu}X_{\mu)} - 2h_{\lambda\mu}X^{\nu} \\
&= 2k_{(\lambda}{}^{\nu}X_{\mu)} + 4k_{(\lambda}{}^{\nu}k_{\mu)}{}^{\alpha}X_{\alpha} \\
&= 2h^{\nu\alpha}\left(h_{(\lambda|\alpha|}X_{\mu)} - \frac{1}{2}(k_{\mu\lambda} + k_{\lambda\mu})X_{\alpha} - 2k_{\alpha(\lambda}k_{\mu)}{}^{\beta}X_{\beta}\right) \\
&= 2h^{\nu\alpha}\left((k_{\mu[\alpha}X_{\lambda]} + k_{\lambda[\alpha}X_{\mu]}) - 2k_{\alpha(\lambda}k_{\mu)\beta}X^{\beta}\right) \\
&= 2h^{\nu\alpha}\left((\delta_{[\alpha}{}^{\beta}X_{\lambda]} - k_{\alpha\lambda}X^{\beta})k_{\mu\beta} + (\delta_{[\alpha}{}^{\beta}X_{\mu]} - k_{\alpha\mu}X^{\beta})k_{\lambda\beta}\right) \\
&= h^{\nu\alpha}\left(S_{\alpha\lambda}{}^{\beta}k_{\mu\beta} + S_{\alpha\mu}{}^{\beta}k_{\lambda\beta}\right) \\
&= 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}.
\end{aligned}$$

Therefore, the Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ of the form (1.16) takes the form (1.7) in virtue of (2.3).

REMARK 2.3. In virtue of Theorem 2.2, we note that under the present condition the ME-connection $\Gamma^{\nu}_{\lambda\mu}$ is of the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}, \tag{2.5}$$

where

$$S_{\lambda\mu}{}^{\nu} = 2\delta_{[\lambda}{}^{\nu}X_{\mu]} - 2k_{\lambda\mu}X^{\nu}, \tag{2.6a}$$

$$U^{\nu}{}_{\lambda\mu} = k_{(\lambda}{}^{\nu}X_{\mu)} + 2k_{(\lambda}{}^{\nu}k_{\mu)}{}^{\alpha}X_{\alpha} = \delta_{(\lambda}{}^{\nu}X_{\mu)} - h_{\lambda\mu}X^{\nu}. \tag{2.6b}$$

In our further considerations, we need a symmetric tensor

$$P_{\lambda\mu} = {}^{(2)}k_{\lambda\mu} - h_{\lambda\mu} \tag{2.7a}$$

and its unique inverse tensor $Q^{\lambda\nu}$ (see [5]) defined by

$$P_{\lambda\mu}Q^{\lambda\nu} = \delta_{\mu}{}^{\nu}. \tag{2.7b}$$

It has been shown that (see [6])

$$Q^{\lambda\nu} = -\frac{1}{g}\sum_{s=0}^{n-1}\widetilde{K}_s{}^{(n-2+\sigma-s)}k^{\lambda\nu}, \tag{2.8a}$$

where

$$\widetilde{K}_s = 1 + K_2 + K_4 + \cdots + K_s. \tag{2.8b}$$

THEOREM 2.4. *Under the present condition, the vector X^ν satisfies*

$$\left(k_{\lambda\nu} - 2\,{}^{(2)}k_{\lambda\nu} + (n-1)h_{\lambda\nu}\right) X^\nu = 0. \tag{2.9}$$

Proof. The relation (2.8) may be obtained by putting $\nu = \mu$ in (2.2a).

THEOREM 2.5. *Under the present condition, the following relation holds in MEX_n*

$$B_{\omega\mu\nu} = -2P_{\nu[\omega}X_{\mu]} + 2k_\omega{}^\alpha P_{\alpha\mu}X_\nu. \tag{2.10}$$

Proof. In view of the process of the proof of Theorem 2.2, it may be easily checked that the condition (2.2a) may be replaced by (1.18). Employing the abbreviations introduced in (1.20) and using the condition (2.1), it follows that

$$\begin{aligned}\overset{pqr}{S}{}_{\omega\mu\nu} &= (h_{\gamma\alpha}X_\beta - h_{\gamma\beta}X_\alpha - 2k_{\alpha\beta}X_\gamma)^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma \\ &= (-1)^r\,{}^{(p+r)}k_{\omega\nu}({}^{(q)}k_\mu{}^\alpha X_\alpha) - (-1)^q\,{}^{(q+r)}k_{\nu\mu}({}^{(p)}k_\omega^\alpha X_\alpha) \\ &\quad - 2(-1)^q\,{}^{(p+q+1)}k_{\omega\mu}({}^{(r)}k_\nu{}^\alpha X_\alpha), \quad (p,q,r = 0,1,2,\cdots).\end{aligned} \tag{2.11}$$

Consequently, using (2.10) the relation (1.18) is reduced to (2.9) as in the following way:

$$\begin{aligned}B_{\omega\mu\nu} &= S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{110}{S}{}_{\omega\mu\nu} \\ &= 2h_{\nu[\omega}X_{\mu]} - 2k_{\omega\mu}X_\nu - {}^{(2)}k_{\omega\nu}X_\mu - k_{\nu\mu}(k_\omega{}^\alpha X_\alpha) \\ &\quad - 2\,{}^{(2)}k_{\omega\mu}(k_\nu{}^\alpha X_\alpha) - k_{\omega\nu}(k_\mu{}^\alpha X_\alpha) + {}^{(2)}k_{\nu\mu}X_\omega \\ &\quad + 2\,{}^{(2)}k_{\omega\mu}(k_\nu{}^\alpha X_\alpha) + k_{\omega\nu}(k_\mu{}^\alpha X_\alpha) + k_{\nu\mu}(k_\omega{}^\alpha X_\alpha) + 2^{(3)}k_{\omega\mu}X_\nu \\ &= 2(h_{\nu[\omega} - {}^{(2)}k_{\nu[\omega})X_{\mu]} + 2({}^{(3)}k_{\omega\mu} - k_{\omega\mu})X_\nu \\ &= -2P_{\nu[\omega}X_{\mu]} + 2k_\omega{}^\alpha P_{\alpha\mu}X_\nu.\end{aligned}$$

III. The ME-vector X^ν.

This section is concerned mainly with an explicit representation of the present condition mentioned in Agreement 2.1 and a general representation of the ME-vector which holds for a general n and all possible classes. Furthermore, we introduce a special kind of representation of X_λ which holds for an even n and for the first class.

In this section we use the following abbreviation for an arbitrary real vector A_λ:

$$^{(p)}A_\lambda = {}^{(p)}k_\lambda{}^\alpha A_\alpha, \quad (p = 0, 1, 2, \cdots). \tag{3.1a}$$

Then, since $^{(p)}k_{\lambda\mu} = (-1)^p \, {}^{(p)}k_{\mu\lambda}$, we have

$$^{(p)}A^\nu = (-1)^p \, {}^{(p)}k_\alpha{}^\nu A^\alpha, \quad (p = 0, 1, 2, \cdots). \tag{3.1b}$$

THEOREM 3.1. *(Recurrence relation) In MEX_n every vector A_ω satisfied the following recurrence relation:*

$$\sum_{s=0}^{n-\sigma} K_s \, {}^{(n-s)}A_\omega = 0. \tag{3.2}$$

Proof. Multiplying A_ν to both side of (1.15) and using the abbreviation (3.1a), we have (3.2).

We use the following quantities:

$$N = \frac{1-n}{2}, \tag{3.3a}$$

$$\widehat{K}_s = \sum_{t=0}^{s} K_t N^{s-t} = N^s + K_2 N^{s-2} + \cdots + K_s, \tag{3.3b}$$

$$Y_\omega = \frac{1}{2} Q^{\nu\mu} B_{\omega\mu\nu}. \tag{3.3c}$$

The quantities introduced in (3.3) are involved in the relations stated in the following two Theorems:

THEOREM 3.2. *We have*

$$\widehat{K}_s = K_s + \widehat{K}_{s-2}N^2. \tag{3.4}$$

THEOREM 3.3. *Under the present condition, the following relations hold in MEX_n:*

$$^{(p)}X_\omega = {}^{(p-1)}Y_\omega + N\,{}^{(p-2)}Y_\omega + N^2\,{}^{(p-2)}X_\omega, \quad (p = 1, 2, 3, \cdots), \tag{3.5}$$

$$^{(p)}Y_\omega = -\frac{1}{4g}\sum_{s=0}^{n-1} K_s \left(\overset{p\hat{s}0}{K}{}_{\omega\alpha\beta} + \overset{p'1\hat{s}}{K}{}_{\omega\alpha\beta} - \overset{p'\hat{s}1}{K}{}_{\omega\alpha\beta} + \overset{1\hat{s}p'}{K}{}_{\alpha\beta\omega}\right) h^{\alpha\beta}, \tag{3.6}$$

where

$$p' = p + 1, \quad \hat{s} = n + \sigma - 2 - s, \quad p = 0, 1, 2, \cdots . \tag{3.7}$$

Proof. The relation (3.4) is a direct consequence of (3.3a) and (3.3b). Multiply $Q^{\nu\mu}$ to both sides of (2.10) and use (2.7) to obtain

$$Q^{\nu\mu}B_{\omega\mu\nu} = (n-1)X_\omega + 2k_\omega{}^\alpha X_\alpha = (n-1)X_\omega + 2\,{}^{(1)}X_\omega.$$

Hence employing the notations introduced in (3.3), we have

$$^{(1)}X_\omega = Y_\omega + NX_\omega. \tag{3.8}$$

Our assertion (3.5) immediately follows from (3.8) as in the following way:

$$\begin{aligned} ^{(p)}X_\omega &= {}^{(p-1)}k_\omega{}^\alpha\,{}^{(1)}X_\alpha = {}^{(p-1)}Y_\omega + N\,{}^{(p-1)}X_\omega \\ &= {}^{(p-1)}Y_\omega + N\,{}^{(p-2)}Y_\omega + N^2\,{}^{(n-2)}X_\omega. \end{aligned}$$

On the other hand, in virtue of (1.19), (1.20), (2.8a), (3.3c), and (3.7), the representation (3.6) may be proved as in the following way:

$$\begin{aligned} ^{(p)}Y_\omega &= \frac{1}{2}Q^{\nu\mu}B_{\gamma\mu\nu}{}^{(p)}k_\omega{}^\gamma \\ &= -\frac{1}{4g}\sum_{s=0}^{n-1}\widetilde{K}_s{}^{(\hat{s})}k^{\nu\mu}\,{}^{(p)}k_\omega{}^\gamma \left(K_{\gamma\mu\nu} + \overset{110}{K}{}_{\gamma\mu\nu} + \overset{011}{K}{}_{\nu\gamma\mu} + \overset{101}{K}{}_{\mu\nu\gamma}\right) \\ &= \text{The right-hand side of (3.6).} \end{aligned}$$

Now, we are ready to prove a general representation of the ME-vector in the folloeing theorem.

THEOREM 3.4. *Under the present condition, the ME-vector X_ω in MEX_n may be given by*

$$(\sigma - 1 - \sigma N)\widehat{K}_{n-\sigma} X_\omega = \sum_{s=0}^{n-\sigma-2} \widehat{K}_s \left({}^{(n-s-1)}Y_\omega + N\, {}^{(n-s-2)}Y_\omega\right) + \sigma \widehat{K}_{n-\sigma} Y_\omega. \tag{3.9}$$

Proof. Substituting (3.5) into (3.2b) with A_ω replaced by X_ω and using (3.3b) and (3.4), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N\, {}^{(n-2)}Y_\omega\right) \\ &\quad + (K_2 + N^2)\,{}^{(n-2)}X_\omega + K_4\,{}^{(n-4)}X_\omega \\ &\quad + \cdots + K_{n-\sigma-2}\,{}^{(\sigma+2)}X_\omega + K_{n-\sigma}\,{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.10a}$$

Substituting again for ${}^{(n-2)}X_\omega$ into (3.10a) from (3.5), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N\, {}^{(n-2)}Y_\omega\right) \\ &+ \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N\, {}^{(n-4)}Y_\omega\right) \\ &+ (K_4 + N^2)\, {}^{(n-4)}X_\omega + K_6\, {}^{(n-6)}X_\omega + \cdots + K_{n-\sigma-2}\, {}^{(\sigma+2)}X_\omega \\ &+ K_{n-\sigma}\, {}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.10b}$$

After $\frac{n-\sigma}{2}$ steps of successive repeated substitutions for ${}^{(p)}X_\omega$, we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N\, {}^{(n-2)}Y_\omega\right) \\ &+ \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N\, {}^{(n-4)}Y_\omega\right) \\ &+ \widehat{K}_4 \left({}^{(n-5)}Y_\omega + N\, {}^{(n-6)}Y_\omega\right) + \cdots + \widehat{K}_2 \left({}^{(\sigma+1)}Y_\omega + N\, {}^{(\sigma)}Y_\omega\right) \\ &+ \widehat{K}_{n-\sigma}\,{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.10c}$$

On the other hand, it follows from (3.1a) and (3.8) that

$${}^{(\sigma)}X_\omega = \sigma Y_\omega + (\sigma N - \sigma + 1) X_\omega. \tag{3.11}$$

Substituting (3.11) into (3.10c), we finally have the relation (3.9).

As a consequence of Theorem 3.4, we have

THEOREM 3.5. *There exists a unique ME-vector (and hence, a unique ME-connection) if, and only if the following condition holds for* $g_{\lambda\mu}$*:*

$$\hat{K}_{n-\sigma} \neq 0. \tag{3.12}$$

Proof. In virtue of (3.9), there exists a unique X_ω if $(\sigma-1-\sigma N)\hat{K}_{n-\sigma} \neq 0$. Hence the relation (3.12) immediately follows since $(\sigma-1-\sigma N) \neq 0$.

REMARK 3.6. (a) In virtue of Theorems 3.4 and 3.5, it is obvious that the present condition holds if, and only if the condition (3.12) holds for the tensor field $g_{\lambda\mu}$.

(b) The representation (3.9) of the ME-vector X_ω is the most general one which holds for a general $n \geq 2$ and for all possible classes.

Now, we present a quite different type of representation of the ME-vector from the general one, which holds in an even-dimensional ME-manifold with a certain special condition imposed on $g_{\lambda\mu}$. We need a tensor $F_{\lambda\mu}$ defined by

$$F_{\lambda\mu} = k_{\lambda\mu} - 2\,{}^{(2)}k_{\lambda\mu}. \tag{3.13}$$

THEOREM 3.7. *The tensor* $F_{\lambda\mu}$ *is of rank* n *if, and only if the tensor field* $g_{\lambda\mu}$ *satisfied the following condition:*

$$\mathfrak{k}\sum_{s=0}^{n-\sigma} 2^s K_s \neq 0. \tag{3.14}$$

Proof. The tensor $F_{\lambda\mu}$ may be written as

$$F_{\lambda\mu} = 2k_{\lambda\alpha}\left(\frac{1}{2}h_{\mu\beta} + k_{\mu\beta}\right)h^{\alpha\beta}. \tag{3.15}$$

Our assertion follows from the following relation which may be obtained from (3.15) and (1.14):

$$\begin{aligned} Det(F_{\lambda\mu}) &= 2^n\mathfrak{k}\left(\mathfrak{h}\sum_{s=0}^{n-\sigma} K_s(\frac{1}{2})^{n-s}\right)\frac{1}{\mathfrak{h}} \\ &= \mathfrak{k}\sum_{s=0}^{n-\sigma} 2^s K_s. \end{aligned}$$

AGREEMENT 3.8. In our further considerations, in this section we restrict ourselves to an even-dimensional ME-manifold MEX_n and use the word " special condition " to descrive the situations that the tensor field $g_{\lambda\mu}$ of an even-dimensional MEX_n satisfies the condition

$$\sum_{s=0}^{n-\sigma} 2^s K_s \neq 0. \tag{3.16}$$

Therefore, under the special condition the tensor $F_{\lambda\mu}$ is of rank n, so that there exists a unique inverse tensor $G^{\lambda\nu}$ defined by

$$G^{\lambda\nu} F_{\lambda\mu} = G^{\nu\lambda} F_{\mu\lambda} = \delta_\mu{}^\nu. \tag{3.17}$$

THEOREM 3.9. *Under the special condition in an even- dimensional MEX_n, the ME-vector X_ω may be given by the following special type of representation:*

$$X_\nu = -\frac{1}{2} G^{\nu\alpha} \partial_\alpha log\ \ g. \tag{3.18}$$

Proof. Multiplying ${}^*g^{\lambda\mu}$, defined by

$${}^*g^{\lambda\nu} g_{\lambda\mu} = {}^*g^{\nu\lambda} g_{\mu\lambda} = \delta_\mu{}^\nu, \tag{3.19}$$

to both sides of (1.5b), we have

$$\partial_\omega log\ \ \mathfrak{g} - 2\Gamma^\alpha_{\alpha\omega} = {}^*g^{\lambda\mu} D_\omega g_{\lambda\mu} = 2S_{\omega\alpha}{}^\alpha. \tag{3.20}$$

On the other hand, multiply $h^{\lambda\mu}$ to both sides of the symmetric part of (1.5b) and make use of (2.1) to obtain

$$\begin{aligned} \partial_\omega log\ \ \mathfrak{h} - 2\Gamma^\alpha_{\alpha\omega} &= h^{\lambda\mu} D_\omega h_{\lambda\mu} = 2h^{\lambda\mu} S_{\omega(\mu}{}^\alpha g_{\lambda)\alpha} \\ &= 2S_{\omega\alpha}{}^\alpha - 2S_{\omega\beta}{}^\alpha k_\alpha{}^\beta \\ &= 2S_{\omega\alpha}{}^\alpha - 4(\delta_{[\omega}{}^\alpha X_{\beta]} - k_{\omega\beta} X^\alpha) k_\alpha{}^\beta \\ &= 2S_{\omega\alpha}{}^\alpha - 2(k_\omega{}^\alpha + 2\,{}^{(2)}k_\omega{}^\alpha) X_\alpha. \end{aligned} \tag{3.21}$$

Subtraction of (3.21) from (3.20) gives in virtue of (1.8)

$$\partial_\omega log\ \ \mathfrak{g} = 2(k_\omega{}^\alpha + 2\,{}^{(2)}k_\omega{}^\alpha) X_\alpha = -2F_{\nu\omega} X^\nu. \tag{3.22}$$

The relation (3.18) immediately follows by multiplying $G^{\lambda\omega}$ to both sides of (3.22) and making use of (3.17).

REMARK 3.10. Incidentally, we note that the present condition (3. 12) and the special condition (3.16) are coincident when $n = 2$. That is

$$1 + 4K \neq 0. \tag{3.23}$$

REMARK 3.11. In virtue of Theorem 3.9, our investigation of the ME-vector under the special condition is reduced to the study of the tensor $G^{\lambda\nu}$. In order to know the ME-vector it is necessary and sufficient to know an explicit representation of $G^{\lambda\nu}$ in terms of $g_{\lambda\mu}$.

In our further considerations, we need the abbreviation ${}^{(p)}X^{\lambda\nu}$ for an arbirtary tensor $X^{\lambda\nu}$ and notations $\overset{\dagger}{K}_s$ defined by

$${}^{(0)}X^{\lambda\mu} = X^{\lambda\mu}, \quad {}^{(p)}X^{\lambda\nu} = {}^{(p)}k^{\lambda}{}_{\alpha}X^{\alpha\nu}, \quad (p = 1, 2, 3, \cdots) \tag{3.24}$$

$$\overset{\dagger}{K}_s = \frac{1}{4}\sum_{t=0}^{s} \frac{1}{2^t} K_{s-t} \tag{3.25}$$

The followings are immediate consequences of (3.22) and (3.23).

$${}^{(q)}k^{\lambda}{}_{\alpha}{}^{(p)}X^{\alpha\nu} = {}^{(p+q)}X^{\lambda\nu}, \tag{3.26}$$

$$\overset{\dagger}{K}_0 = \frac{1}{4}, \quad \overset{\dagger}{K}_2 = \frac{1}{4}(K_2 + \frac{1}{4}), \quad \overset{\dagger}{K}_4 = \frac{1}{4}(K_4 + \frac{1}{4}K_2 + \frac{1}{16}), \quad \cdots, \tag{3.27a}$$

$$\overset{\dagger}{K}_s = \frac{1}{4}(K_s + \overset{\dagger}{K}_{s-2}). \tag{3.27b}$$

THEOREM 3.12. *(Recurrence relation) In an even-dimensional MEX_n, the tensor ${}^{(p)}G^{\lambda\nu}$ satisfies the following recurrence relation:*

$$\sum_{s=0}^{n} K_s{}^{(n-s)}G^{\lambda\nu} = 0, \tag{3.28a}$$

or equivalently

$${}^{(n)}G^{\lambda\nu} + K_2{}^{(n-2)}G^{\lambda\nu} + \cdots + K_{n-2}{}^{(2)}G^{\lambda\nu} + K_n G^{\lambda\nu} = 0. \tag{3.28b}$$

Proof. Multiplying $G^{\lambda\mu}$ to both sides of (1.15) and making use of (3.24), we have (3.28). Note that $n - s$ is even, so that ${}^{(n-s)}k_{\lambda\mu}$ is symmetric.

THEOREM 3.13. *Under the special condition, the following relations hold in MEX_n:*

$$ {}^{(p+2)}G^{\lambda\nu} + \frac{1}{2}{}^{(p+1)}G^{\lambda\nu} + \frac{1}{2}{}^{(p)}k^{\lambda\nu} = 0, \quad (p = 0, 1, 2, \cdots), \tag{3.29} $$

$$ \begin{aligned} &{}^{(q)}G^{\lambda\nu} \\ =&\frac{1}{4}{}^{(q-2)}G^{\lambda\nu} - \frac{1}{2}{}^{(q-2)}k^{\lambda\nu} + \frac{1}{4}{}^{(q-3)}k^{\lambda\nu}, \quad (q = 3, 4, 5, \cdots). \end{aligned} \tag{3.30} $$

Proof. Substitution of (3.13) into (3.17) gives

$$ 2\,{}^{(2)}G^{\lambda\mu} + {}^{(1)}G^{\lambda\mu} + h^{\lambda\mu} = 0. \tag{3.31} $$

The relation (3.29) may be obtained by multiplying $\frac{1}{2}{}^{(p)}k^{\nu}{}_{\lambda}$ to both sides of (3.31). Using (3.29) twice, we have (3.30).

Now, we are ready to prove the following Theorem, which present a representation of the $G^{\lambda\mu}$ under the special condition.

THEOREM 3.14. *Under the special condition in an even-dimensional MEX_n, the tensor $G^{\lambda\nu}$ may be given by*

$$ 2k\overset{\dagger}{K}_n G^{\lambda\nu} = \overset{\dagger}{K}_{n-2}(kh^{\lambda\nu} + 2\overset{\dagger}{K}_{n-2}k^{\lambda\nu}) - 2\overset{\dagger}{K}_n\Lambda^{\lambda\nu} - \overset{\dagger}{K}_{n-2}{}^{(1)}\Lambda_{\lambda\nu}, \tag{3.32} $$

where

$$ \Lambda^{\lambda\nu} = \sum_{s=0}^{n-4} \overset{\dagger}{K}_s \left(-2\,{}^{(n-s-2)}k^{\lambda\nu} + {}^{(n-3-s)}k^{\lambda\nu}\right). \tag{3.33} $$

Proof. Substituting (3.30) into (3.28b) for ${}^{(n)}G^{\lambda\nu}$ and making use of (3.27), we have

$$ \begin{aligned} &\overset{\dagger}{K}_0(-2\,{}^{(n-2)}k^{\lambda\nu} + {}^{(n-3)}k^{\lambda\nu}) + 4\overset{\dagger}{K}_4{}^{(n-4)}G^{\lambda\nu} \\ &+ \cdots + K_{n-2}{}^{(2)}G^{\lambda\nu} + K_nG^{\lambda\nu} = 0. \end{aligned} \tag{3.34a} $$

Substituting again for ${}^{(n-2)}G^{\lambda\nu}$ into (3.34a) from (3.30) gives

$$\begin{aligned} &\overset{\dagger}{K}_0(-2\,{}^{(n-2)}k^{\lambda\nu} + {}^{(n-3)}k^{\lambda\nu}) + \overset{\dagger}{K}_2(-2\,{}^{(n-4)}k^{\lambda\nu} + {}^{(n-5)}k^{\lambda\nu}) \\ &\quad + 4\overset{\dagger}{K}_4{}^{(n-4)}G^{\lambda\nu} + \cdots + K_{n-2}{}^{(2)}G^{\lambda\nu} + K_nG^{\lambda\nu} = 0. \end{aligned} \tag{3.34b}$$

After $\frac{n-2}{2}$ steps of successive repeated substitution for ${}^{(q)}G^{\lambda\nu}$, we have in virtue of (3.33)

$$\Lambda^{\lambda\nu} + 4\overset{\dagger}{K}_{n-4}{}^{(2)}G^{\lambda\nu} + K_nG^{\lambda\nu} = 0. \tag{3.34c}$$

Comparison of (3.34) with $A^{(2)}G^{\lambda\nu} + BG^{\lambda\nu} + \Lambda^{\lambda\nu} = 0$ gives

$$A = 4\overset{\dagger}{K}_{n-2}, \quad B = K_n = k. \tag{3.35}$$

Consequently, the relation (3.32) follows by substituting (3.35) into $B(A+4B)G^{\lambda\nu} = 2ABh^{\lambda\nu} + A^2k^{\lambda\nu} - (A+4B)\Lambda^{\lambda\nu} - 2A\,{}^{(1)}\Lambda^{\lambda\nu}$ and making use of (3.37b).

Now that we have obtained a representation of $G^{\lambda\nu}$ in Theorem 3.14, under the special condition it is possible for us to represent the ME-vector X^ν in terms of $g_{\lambda\mu}$ by only substituting (3.32) into (3.18).

THEOREM 3.15. *Under the special condition in an even-dimensional MEX_n, the ME-vector X^ν may be given by*

$$\begin{aligned} &4k\overset{\dagger}{K}_nX^\nu \\ &= -\left(\overset{\dagger}{K}_{n-2}(kh^{\nu\alpha} + 2\overset{\dagger}{K}_{n-2}k^{\nu\alpha}) - 2\overset{\dagger}{K}_n\Lambda^{\nu\alpha} - \overset{\dagger}{K}_{n-2}\,{}^{(1)}\Lambda^{\nu\alpha}\right)\partial_\alpha(\log\ g). \end{aligned} \tag{3.36}$$

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