Bull. Korean Math. Soc. **31** (1994), No. 2, pp. 237–242

CHARACTERIZATIONS OF PARTITION LATTICES

YOUNG-JIN YOON

I. Introduction

One of the most well-known geometric lattices is a partition lattice. Every upper interval of a partition lattice is a partition lattice. The Whitney numbers of the partition lattices are the Stirling numbers, and the characteristic polynomial is a falling factorial. The set of partitions with a single non-trivial block containing a fixed element is a Boolean sublattice of modular elements, so the partition lattice is supersolvable in the sense of Stanley [6].

In this paper, we rephrase four results due to Heller [1] and Murty [4] in terms of matroids and give several characterizations of partition lattices.

Our notation and terminology follow those in [8, 9]. To clarify our terminology, let G be a finite geometric lattice. If S is the set of points (or rank-one flats) in G, the lattice structure of G induces the structure of a (combinatorial) geometry, also denoted by G, on S. The *size* $|G|$ of the geometry G is the number of points in G. Let T be a subset of S. The *deletion* of T from G is the geometry on the point set $S \backslash T$ obtained by restricting G to the subset $S \backslash T$. The *contraction* G/T of G by T is the geometry induced by the geometric lattice $[cl(T), \hat{1}]$ on the set S' of all flats in G covering $cl(T)$. (Here, $cl(T)$ is the closure of T, and $\hat{1}$ is the maximum of the lattice G.) Thus, by definition, the contraction of a geometry is always a geometry. A geometry which can be obtained from G by deletions and contractions is called a *minor* of G.

Received May 3, 1993.

2. Preliminaries

Let S be a finite set of n elements. A partition π of S is a family of disjoint subsets $B_1, B_2, \dots, B_k$, called blocks, whose union is S. There is a natural ordering of partitions, which is defined as follows : $\pi \leq \sigma$ whenever every block of a partition π is contained in a block of a partition σ. Denote the lattice of partitions of a set with n elements by P_n. We call P_n the *partition lattice* of rank $n-1$.

Let G be a geometry. Then we can associate with G a geometric lattice $L(G)$ whose elements are the flats of G ordered by inclusion. Note that the partition lattice P_n is isomorphic to the lattice of flats of the polygon matroid of the complete graph K_n.

THEOREM 1 [1]. *Let G be a binary geometry of rank n and let a be a point in G. If $|G| - |G/a| > n$, then G contains the Fano plane as a minor.*

COROLLARY 1 [1, 4]. *A binary rank-n geometry not containing the Fano plane as a minor contains at most $\binom{n+1}{2}$ points.*

As a contrapositive of Corollary 1, we have the following.

COROLLARY 2 [1, 4]. *If a binary geometry of rank n has more than $\binom{n+1}{2}$ points, then it contains the Fano plane as a minor.*

THEOREM 2 [1, 4]. *If a binary rank-n geometry G not containing the Fano plane as a minor contains $\binom{n+1}{2}$ points, then G is the polygon matroid of the complete graph K_{n+1} ; that is, $L(G) \cong P_{n+1}$.*

THEOREM 3. *If a geometry G has $\binom{n+1}{2}$ points and $L(G/a) \cong P_n$ for every point a in G, then $L(G) \cong P_{n+1}$.*

Proof. Note that G has rank n. Since $L(G/a) \cong P_n$ for every point a in G, the scum theorem [9, p.240] implies that G is binary. For $n = 1$ and $n = 2$, the theorem is true.

Let $n = 3$. If G contains the Fano plane as a minor, then G is isomorphic to the Fano plane. But G and the Fano plane have a different number of points, so we have a contradiction. Thus G cannot contain the Fano plane as a minor. By Theorem 2, we have $L(G) \cong P_4$.

Let $n \geq 4$. If G contains the Fano plane as a minor, then by the scum theorem G/a contains the Fano plane as a minor for some point a in G. Since $L(G/a) \cong P_n$, the polygon matroid of the complete graph K_n contains the Fano plane as a minor. But this is contradictory to [10, Theorem 1.5.4.]. Thus G cannot contain the Fano plane as a minor. By Theorem 2, we have $L(G) \cong P_{n+1}$.

Kahn and Kung [3] defined splitting in geometries. Let G be a geometry. Then G *splits* if G is the union of two of its proper flats. G is said to be *non-splitting* otherwise. We shall be more concerned with non-splitting geometries. An example of a non-splitting geometry is $M(K_n)$, the polygon matroid of the complete graph K_n on n vertices.

A geometry G is said to be *upper homogeneous* if for $k = 1, 2, \ldots, r(G)$, $G/x \cong G/y$ for every pairs x, y of flats of rank k.

LEMMA 1. *If a geometry G is upper homogeneous, has a modular copoint, and $|G| > r(G)$, then G is non-splitting.*

Proof. It suffices to show that if a geometry G is upper homogeneous and has a modular copoint and is splitting, then $|G| = r(G)$.

Use induction on $n = r(G)$. For $n = 1$, 2, the lemma is true. Assume it holds for a geometry of rank less than n. Let a be a point in G.

Suppose that G/a is non-splitting. Since G is splitting, we have $G = A \cup B$ where A and B are proper flats of G. Assume to the contrary that $A \cap B = \emptyset$. Let x be a modular copoint of G. Then x is splitting, i.e. $x = (A \cap x) \cup (B \cap x)$ where $A \cap x$ and $B \cap x$ are proper flats of x. Since x is isomorphic to G/a and G/a is non-splitting, we have a contradiction. Thus $A \cap B \neq \emptyset$. Since $G/c = (A/c) \cup (B/c)$ for a point c in $A \cap B$, it follows that G/c is splitting. Since G is upper homogeneous, we have a contradiction. Thus G/a is splitting.

Note that G/a is upper homogeneous and has a modular copoint. By the induction hypothesis, we have $|G/a| = r(G/a)$, i.e. G/a is a Boolean algebra. Since G/a is a Boolean algebra for every point a in G, Theorem [2, p.89] and the scum theorem implies that G is a Boolean algebra, i.e. $|G| = r(G)$.

LEMMA 2. *The following statements are equivalent.*

(1) *G is non-splitting*
(2) *If x is a copoint of G, then $G\backslash x$ contains a basis.*

Proof. (1) $\Longrightarrow$ (2) ; Suppose that $G\backslash x$ does not contain a basis. Then $cl(G\backslash x)$ is a proper flat and $G = x \cup cl(G\backslash x)$.

(2) $\Longrightarrow$ (1) ; Suppose that G is splitting. Let $G = A \cup B$ where A and B are proper flats of G. Let α be a copoint of A containing $A \cap B$ and let $x = \alpha \cup B$ be a copoint of G. Then $G\backslash x$ is contained in A. Since A is a proper flat, A does not contain a basis. Thus $G\backslash x$ does not contain a basis.

Stonesifer and Bogart [7] proved the following theorem in terms of geometric lattices. Here we prove this by the previous results.

THEOREM 4. *If a geometry G has a modular copoint and $L(G/a) \cong P_n$ for every point a in G, then $L(G) \cong P_{n+1}$ for $n \geq 4$.*

Proof. . By Theorem 3, it suffices to show that $|G| = \binom{n+1}{2}$. Let x be a modular copoint in G. Then the interval $[0, x]$ is isomorphic to $L(G/a)$ for a point a not in x. Thus $|x| = |G/a| = |P_n| = \binom{n}{2}$. Since G contains no 4-point line as a minor by the scum theorem, G is binary. Also, since x is a modular copoint, no 2-point line is contained in $G\backslash x$. If a line ℓ is not in x, then $r(x \wedge \ell) = r(x) + r(\ell) - r(x \vee \ell) = (n-1) + 2 - n = 1$. It implies that every 3-point line (in G) not in x contains one point in x.

Let $|G\backslash x| = k$. Since G has a modular copoint and is upper homogeneous and $|G| > |G/a| = \binom{n}{2} \geq n$ for $n \geq 4$, by Lemma 1 and Lemma 2, we have $k \geq n$. If $|G| - |G/a| = k > n$, then Theorem 1 implies that G contains the Fano plane as a minor. But for $n \geq 4$, G cannot contain the Fano plane as a minor by the scum theorem. Thus $k = n$ and $|G| = \binom{n+1}{2}$.

Let $p(G; \lambda)$ be the characteristic polynomial of a geometry G. Then we have

$$p(P_{n+1} : \lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n).$$

In the next section, we characterize the partition lattice in terms of its characteristic polynomial and some additional conditions.

3. Main theorems

THEOREM 5. *If a geometry G is upper homogeneous, has a modular copoint, and $p(G;\lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n)$, then $L(G) \cong P_{n+1}$.*

Proof. Induction on n. For $n = 1$, 2, the theorem is true. Assume that it holds for a geometry of rank less than n.

Let x be a modular copoint of G. By Lemma 1 and Lemma 2, we have $|G\backslash x| \geq n$. Since x is modular, by the modular factorization theorem [5, Theorem 2] , we have $|G\backslash x| \leq n$. Thus $|G\backslash x| = n$ and $p(x;\lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n + 1)$. Since $[0, x]$ is isomorphic to $L(G/a)$ for some point a not in x and G is upper homogeneous, we have $p(G/a;\lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n + 1)$ for every point a in G. Note that G/a is upper homogeneous for every point a in G. Since x/b is a modular copoint of G/b for a point b in x, it follows that G/a has a modular copoint for every point a in G. By the induction hypothesis, $L(G/a) \cong P_n$ for every point a in G. Thus Theorem 4 implies $L(G) \cong P_{n+1}$.

THEOREM 6. *If a geometry G is non-splitting, supersolvable, and $p(G;\lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n)$, then $L(G) \cong P_{n+1}$.*

Proof. We induct on the rank n of a geometry G. For $n \leq 3$, the theorem is true. Assume it holds for a geometry of rank less than n.

Let x be a modular copoint in a maximal chain of flats. Then we have $|G\backslash x| \leq n$. Since G is non-splitting, Lemma 2 gives $|G\backslash x| \geq n$. Thus $|G\backslash x| = n$. By the modular factorization theorem, we have $p(x;\lambda) = (\lambda - 1)(\lambda - 2)\dots(\lambda - n + 1)$. Suppose that x is splitting. Let $x = A \cup B$ where A and B are proper flats of x. Since $L(G/a) \cong [0, x]$ for a point a not in x, it follows that $G \cong cl(A \cup a) \cup cl(B \cup a)$ and so G is splitting, a contradiction. Thus x is non-splitting.

Now x satisfies all the conditions of the theorem. By the induction hypothesis, we have $L(x) \cong P_n$. Note that $G \backslash x$ is exactly a basis of G. Since x is a modular copoint with $|x| = \binom{n}{2}$, every two points of $G \backslash x$ is connected to a unique point in x by a 3-point line. Therefore $L(G) \cong P_{n+1}$.

ACKNOWLEDGEMENT. This paper is based on my Ph.D. thesis at University of North Texas. I would like to thank Professor Joseph P. S. Kung for his encouragement and helpful suggestions.

References

1. I. Heller, *On linear systems with integral valued solutions*, Pacific J. Math. **7** (1957), 1351-1364.
2. C. Greene, *Lectures on combinatorial geometries*, Notes taken by D. Kennedy from the National Science Seminar on Combinatorial Theory, Bowdoin College, not published.
3. J. Kahn and J. P. S. Kung, *Varieties of combinatorial geometries*, Trans. Amer. Math. Soc. **271** (1982), 485-499.
4. U. S. R. Murty, *Extremal matroids with forbidden restrictions and minors (synopsis)*, Proceedings of the seventh southeastern conference on combinatorics, graph theory, and computing, Louisiana State University, Baton Rouge, LA, 1976, Congressus Numerantium, No. 17, Utilitas Math., Winnepeg, Manitoba 1976 pp. 463-468.
5. R. P. Stanley, *Modular elements of geometric lattices*, Algebra Universalis **1** (1971), 214-217.
6. ———, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197-217.
7. J. R. Stonesifer and K. P. Bogart, *Characterizations of partition lattices*, Algebra Universalis **19** (1984), 92-98.
8. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.
9. N. L. White (ed.), *Theory of Matroids*, Cambridge University Press, Cambridge, 1986.
10. ———, *Combinatorial Geometries*, Cambridge University Press, Cambridge, 1987.
11. Young-jin Yoon, *Characterizations of some combinatorial geometries*, Ph.D. dissertation, University of North Texas, 1992.

DEPARTMENT OF MATHEMATICS, KUNSAN NATIONAL UNIVERSITY, KUNSAN, CHUN BUK 573-360, KOREA