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GENERIC SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF A SASAKIAN SPACE FORM

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Dedicated to Professor Tsunero Takahashion his 60th birthday

0. Introduction

The theory of a submanifold of a Sasakian manifold was investigated from two different points of view, one is the case where submanifolds are tangent to the structure vector, and the other is the case where those are normal to the structure vector [9], [11], [12] etc. Many subjects for such submanifolds of a Sasakian space form have been studied in [1], [2], [4], [5], [6], [14] and so on. But, without these considerations a generic submanifold of a Sasakian manifold are defined as follows : Let M be a submanifold of a Sasakian manifold $\tilde{M}$ with almost contact metric structure (ϕ, G, V). If each normal space is mapped into the tangent space under the action of ϕ, M is called a generic submanifold of $\tilde{M}$ [3], [9]. For example, hypersurfaces of a Sasakian manifold are generic.

The purpose of the present paper is to study generic submanifolds of a Sasakian space form with nonvanishing parallel mean curvature vector field such that the shape operator in the direction of the mean curvature vector field commutes with the structure tensor field induced on the submanifold.

In §1 we state general formulas on generic submanifolds of a Sasakian manifold, especially those of a Sasakian space form. §2 is devoted to the study a generic submanifold of a Sasakian manifold, which is not tangent to the structure vector. In §3 we investigate generic submanifolds, not tangent to the structure vector, of a Sasakian space form

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with nonvanishing parallel mean curvature vector field. In §4 we discuss generic submanifolds tangent to the structure vector of a Sasakian space form and compute the restricted Laplacian for the shape operator in the direction of the mean curvature vector field. As applications of these, in the last §5 we prove our main results.

1. Generic submanifolds of a Sasakian manifold

In this section, the fundamental properties of generic submanifolds of a Sasakian manifold are recalled [3], [9].

Let $\tilde{M}$ be a Sasakian manifold of dimension $2m+1$ with almost contact metric structure (ϕ, G, V). Then for any vector fields X and Y on $\tilde{M}$, we have

$$\phi^2 X = -X + v(X)V,\ G(\phi X, \phi Y) = G(X,Y) - v(X)v(Y),$$
$$v(\phi X) = 0,\ \phi V = 0, v(V) = 1,\ G(X,V) = v(X).$$

Since $\tilde{M}$ is a Sasakian manifold, we then have

$$\tilde{\nabla}_X V = \phi X, \quad (\tilde{\nabla}_X \phi)Y = -G(X,Y)V + v(Y)X, \tag{1.1}$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be an (n+1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \longrightarrow \tilde{M}$. When the argument is local, M need not be distinguished from $i(M)$ itself. Throughout this paper the indices $i, j, k, \cdots$ run from 1 to $n+1$. We represent the immersion i locally by

$$y^A = y^A(x^h), \quad (A = 1, \cdots, n+1, \cdots, 2m+1)$$

and put $B_j{}^A = \partial_j y^A$, $(\partial_j = \frac{\partial}{\partial x^j})$ then $B_j = (B_j{}^A)$ are (n+1)-linearly independent local tangent vector fields of M. We choose $2m-n$ mutually orthogonal unit normals $C_x = (C_x{}^A)$ to M. Hereafter the indices $u, v, w, x, \cdots$ run from $n+2$ to $2m+1$ and the summation convention will be used. The immersion being isometric, the induced Riemannian metric tensor g with components g_{ji} and the metric tensor δ with components δ_{yx} of the normal bundle are respectively obtained :

$$g_{ji} = G(B_j, B_i), \quad \delta_{yx} = G(C_y, C_x).$$

By denoting ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\nabla_j B_i = A_{ji}{}^x C_x, \quad \nabla_j C_x = -A_j{}^h{}_x B_h, \tag{1.2}$$

where $A_{ji}{}^x$ are components of the second fundamental tensors and the shape operator A^x in the direction of C_x are related by

$$A^x = (A_j{}^{hx}) = (A_{jiy} g^{ih} \delta^{yx}), \quad g^{ji} = (g_{ji})^{-1}.$$

An $(n+1)$-dimensional submanifold M of a Sasakian manifold $\tilde{M}$ is called a *generic submanifold* if

$$\phi N_p(M) \subset T_p(M)$$

at each point $p \in M$, where $T_p(M)$ is the tangent space of M at p and $N_p(M)$ the normal space at p, [3], [9].

From now on, we have only to consider generic submanifolds of a Sasakian manifold. Then the transforms of B_i and C_x by ϕ are respectively represented in each coordinate neighborhood as follows :

$$\phi B_j = f_j{}^h B_h - J_j{}^x C_x, \quad \phi C_x = J_x{}^h B_h, \tag{1.3}$$

where we have put $f_{ji} = G(\phi B_j, B_i)$, $J_{jx} = -G(\phi B_j, C_x)$, $J_{xj} = G(\phi C_x, B_j)$, $f_j{}^h = f_{ji} g^{ih}$ and $J_j{}^x = J_{jy}\delta^{yx}$. From these definitions we verify that $f_{ji} + f_{ij} = 0$ and $J_{jx} = J_{xj}$.

Also, we can put the Sasakian structure vector V of the form

$$V = \xi^h B_h + w^x C_x, \tag{1.4}$$

where $\xi_i = G(B_i, V)$ and $w_x = G(C_x, V)$, ξ^h being the associated vector with ξ_i.

By the properties of the Sasakian structure tensor, it follows from (1.3) and (1.4) that we obtain

$$f_j{}^t f_t{}^h = -\delta_j{}^h + \xi_j \xi^h + J_j{}^x J_x{}^h, \tag{1.5}$$

$$J_y{}^t J_t{}^x = \delta_y{}^x - w_y w^x, \tag{1.6}$$

$$f_t{}^h J_x{}^t = w_x \xi^h, \quad f_{jt}\xi^t = J_{jx} w^x, \tag{1.7}$$

$$J_t{}^x \xi^t = 0, \tag{1.8}$$

$$w_x w^x = 1 - \xi_t \xi^t. \tag{1.9}$$

Differentiating (1.3) and (1.4) covariantly along M and making use of (1.1), (1.2) and these equations, we easily find (cf. [3])

$$\nabla_k f_j{}^h = -g_{kj}\xi^h + \delta_k{}^h \xi_j + A_{kj}{}^x J_x{}^h - A_k{}^{hx} J_{jx}, \tag{1.10}$$

$$\nabla_k J_{jx} = g_{kj} w_x + A_{krx} f_j{}^r, \tag{1.11}$$

$$A_{jrx} J^{ry} = A_j{}^{ry} J_{rx}, \tag{1.12}$$

$$\nabla_j \xi_i = f_{ji} + A_{jix} w^x, \tag{1.13}$$

$$\nabla_j w_x = -J_{jx} - A_{jrx}\xi^r. \tag{1.14}$$

In the rest of this section we suppose that the ambient Sasakian manifold $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature c and of real dimension $2m+1$, which is called a *Sasakian space form*, and is denoted by $\tilde{M}^{2m+1}(c)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned} &\tilde{R}_{DCBA} \\ =&\frac{1}{4}(c+3)(G_{DA}G_{CB} - G_{DB}G_{CA}) \\ &+ \frac{1}{4}(c-1)(V_C V_A G_{DB} - V_C V_B G_{DA} + V_D V_B G_{CA} - V_D V_A G_{CB} \\ &+ \phi_{DA}\phi_{CB} - \phi_{DB}\phi_{CA} - 2\phi_{DC}\phi_{BA}). \end{aligned}$$

Thus, we see, using (1.3) and (1.4), that equations of the Gauss, Codazzi and Ricci for M are respectively obtained :

$$\begin{aligned}R_{kjih} =&\frac{1}{4}(c+3)(g_{kh}g_{ji}-g_{jh}g_{ki})+A_{kh}{}^{x}A_{jix}-A_{jh}{}^{x}A_{kix}\\&+\frac{1}{4}(c-1)(\xi_k\xi_i g_{jh}-\xi_j\xi_i g_{kh}+\xi_j\xi_h g_{ki}-\xi_k\xi_h g_{ji}\\&+f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{kj}f_{ih}),\end{aligned}\tag{1.15}$$

$$\begin{aligned}\nabla_k A_{ji}{}^{x}-\nabla_j A_{ki}{}^{x} =&\frac{1}{4}(c-1)\{J_j{}^{x}f_{ki}-J_k{}^{x}f_{ji}+2J_i{}^{x}f_{kj}\\&+w^{x}(\xi_j g_{ki}-\xi_k g_{ji})\},\end{aligned}\tag{1.16}$$

$$R_{jixy}=\frac{1}{4}(c-1)(J_{jy}J_{ix}-J_{jx}J_{iy})+A_{jry}A_i{}^{r}{}_{x}-A_{iry}A_j{}^{r}{}_{x},\tag{1.17}$$

where R_{kjih} and R_{jixy} are components of the Riemannian curvature tensor of M and that with respect to the connection induced in the normal bundle of M respectively. We verify from (1.15) that the Ricci tensor S with components S_{ji} of M can be expressed as follows :

$$\begin{aligned}S_{ji} =&\frac{1}{4}\{(c+3)n+(2+w_x w^{x})(c-1)\}g_{ji}\\&-\frac{1}{4}(c-1)\{(n+2)\xi_j\xi_i+3J_j{}^{z}J_{iz}\}+h^{x}A_{jix}-A_j{}^{rx}A_{irx}\end{aligned}\tag{1.18}$$

with the aid of (1.5), where $h^x = \mathrm{Tr}A^x$.

2. Normal parts of the structure vector

Let M be a generic submanifold of a Sasakian manifold and denote by $\alpha = w_x w^x$. In the sequel the index $n+2$ will be denoted by the symbol $*$. Suppose that the function α does not vanish almost everywhere, and that $A^* f = fA^*$, namely

$$A_{jr}{}^{*}f_i{}^{r}+A_{ir}{}^{*}f_j{}^{r}=0\tag{2.1}$$

holds on M. By transvecting $f_k{}^i$ and making use of (1.5), we then have

$$A_{jk}{}^* - (A_{jr}{}^* J_z{}^r) J_k{}^z - (A_{jr}{}^* \xi^r)\xi_k - A_{sr}{}^* f_j{}^s f_k{}^r = 0,$$

and hence taking the skew-symmetric part with respect to k and j,

$$(A_{jr*} J_z{}^r) J_k{}^z - (A_{kr*} J_z{}^r) J_j{}^z + (A_{jr*}\xi^r)\xi_k - (A_{kr*}\xi^r)\xi_j = 0. \quad (2.2)$$

If we transvect this by ξ^k and make use of (1.8) and (1.9), then we get

$$(1-\alpha)\{A_{jr*}\xi^r - \lambda_*\xi_j\} - (A_{sr*}\xi^s J_z{}^r) J_j{}^z = 0, \quad (2.3)$$

where we have put $(1-\alpha)\lambda_x = A_{jix}\xi^j\xi^i$. Transvecting this with $J_y{}^j$ and using (1.6) and (1.8), we find

$$\alpha A_{sr*}\xi^r J_y{}^s - w_y(A_{sr*}\xi^s J_z{}^r w^z) = 0,$$

which joined with (1.7) gives

$$\alpha A_{sr*}\xi^r J_y{}^s + w_y(A_{jr*} f_i{}^r \xi^j \xi^i) = 0.$$

Thus it is, taking account of (2.1), clear that $A_{sr*}\xi^r J_y{}^s = 0$ because the function α is not vanish almost everywhere. Hence (2.3) implies $(1-\alpha)(A_{jr*}\xi^r - \lambda_*\xi_j) = 0$ and consequently

$$A_{jr*}\xi^r = \lambda_*\xi_j \quad (2.4)$$

with the aid of (1.9). Therefore (2.2) is reduced to

$$(A_{jr*} J_z{}^r) J_i{}^z - (A_{ir*} J_z{}^r) J_j{}^z = 0. \quad (2.5)$$

Because of (1.7), (2.1) and (2.4), we have

$$A_{jr*} J_z{}^r w^z = -A_{jr*} f_s{}^r \xi^s = \lambda_* w_z J_j{}^z.$$

Thus, by transvecting $J_w{}^i$ to (2.5) and using (1.6), we find

$$A_{jr*} J_x{}^r = Q_{xz*} J_j{}^z, \quad (2.6)$$

where we have put

$$Q_{xz*} = \lambda_* w_x w_z + P_{xz*}, \quad P_{xyz} = A_{jix} J_y{}^j J_z{}^i. \quad (2.7)$$

We notice here that P_{xyz} is symmetric for all indices because of (1.12). First of all, we prove

LEMMA 2.1. *Let M be a generic submanifold satisfying (2.1) of a Sasakian manifold. If the function α does not vanish almost everywhere, then we have*

$$A_{ji*} = \lambda_*(g_{ji} - J_j{}^z J_{iz}) + Q_{yz*} J_j{}^y J_i{}^z. \tag{2.8}$$

Proof. From (1.14) and (2.4) we have

$$-\nabla_j w_* = J_{j*} + \lambda_* \xi_j.$$

Differentiating this covariantly along M and substituting (1.11) and (1.13), we find

$$-\nabla_k \nabla_j w_* = g_{kj} w_* + A_{kr*} f_j{}^r + (\nabla_k \lambda_*)\xi_j + \lambda_*(f_{kj} + A_{kjx} w^x),$$

from which taking the skew-symmetric part and making use of (2.1),

$$2(A_{kr*} f_j{}^r + \lambda_* f_{kj}) + \xi_j \nabla_k \lambda_* - \xi_k \nabla_j \lambda_* = 0. \tag{2.9}$$

If we transvect (2.9) with ξ^k and use (2.4), then we get

$$(1-\alpha)(\nabla_j \lambda_* - \mu \xi_j) = 0, \tag{2.10}$$

where we have put $(1-\alpha)\mu = \xi^t \nabla_t \lambda_*$. Thus, (2.9) turns out to be

$$A_{kr*} f_j{}^r = \lambda_* f_{jk} \tag{2.11}$$

because of (1.9) and (1.13). By transvecting $f_i{}^j$ and taking account of (1.5), (2.4) and (2.6), we see that (2.8) is valid. This completes the proof of Lemma 2.1.

Differentiating (2.4) covariantly and using (1.13) and (2.11), we obtain

$$(\nabla_k A_{jr*})\xi^r + w^x A_{krx} A_j{}^{r*} = (\nabla_k \lambda_*)\xi_j + \lambda_* w^x A_{kjx}, \tag{2.12}$$

from which, transforming $f_l{}^j$ and making use of (2.11),

$$(\nabla_k A_{rs*})\xi^r f_l{}^s = (\nabla_k \lambda_*) f_{ls} \xi^s. \tag{2.13}$$

LEMMA 2.2. *Under the same assumptions as that in Lemma 2.1, we have*

$$Q_{xz*}w^z = \lambda_* w_x. \tag{2.14}$$

Proof. Transforming (2.6) by $f_i{}^j$ and taking account of (1.7), (2.1) and (2.4), we find

$$(\lambda_* w_x - Q_{xz*}w^z)\xi_j = 0.$$

Let M_0 be a set of points in M at which $\|\lambda_* w_x - Q_{xz*}w^z\| \neq 0$. Then we have $\xi_j = 0$ and hence $A_{jix}w^x = 0$ on M_0 because of (1.13). Accordingly we have $\alpha = 1$ and $P_{yz*}w^z = 0$, which produce a contradiction because of (2.7). Thus we have (2.14) on M. This completes the proof.

3. Nonvanishing parallel mean curvature vector field

In this section, we consider that a generic submanifold M of a Sasakian space form $\tilde{M}^{2m+1}(c)$.

Let H be a mean curvature vector of a generic submanifold M. Namely, it is defined by

$$H = g^{ji}A_{ji}{}^x C_x/(n+1) = h^x C_x/(n+1),$$

which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

In what follows we suppose that the mean curvature vector H of M is nonzero and is parallel in the normal bundle. Then we may choose a local filed $\{C_x\}$ in such a way that $H = \sigma C_{n+2} = \sigma C_*$, where $\sigma = \|H\|$ is nonzero constant. Because of the choice of the local field, the parallelism of H yields

$$\begin{cases} h^x = 0, \quad x \geq n+3 \\ h^* = (n+1)\sigma. \end{cases} \tag{3.1}$$

Here we notice that the condition (2.1) does not depend on the choice of the local field since we have (3.1).

The parallelism of the mean curvature vector yields that $R_{jix*} = 0$ because M is not minimal. Thus (1.17) implies

$$A_{jr}{}^{x}A_{i}{}^{r*} - A_{ir}{}^{x}A_{j}{}^{r*} = \frac{1}{4}(c-1)(J_{i}{}^{x}J_{j}{}^{*} - J_{j}{}^{x}J_{i}{}^{*}) \tag{3.2}$$

with the aid of (3.1).

LEMMA 3.1. *Let M be a generic submanifold satisfying (2.1) of $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If the function $\alpha(1-\alpha)$ does not vanish almost everywhere, then λ_* is constant.*

Proof. Differentiating (2.11) covariantly and using (1.10), (2.4) and (2.6), we find

$$\begin{aligned}&(\nabla_k A_{ir*})f_j{}^r + (A_{ki*} - \lambda_* g_{ki})\xi_j \\ =&(A_{ir*}A_k{}^{rx} - \lambda_* A_{ik}{}^{x})J_{jx} - \theta_{ix}A_{kj}{}^{x} + (\nabla_k\lambda_*)f_{ji},\end{aligned} \tag{3.3}$$

where we have put

$$\theta_{ix} = Q_{xz*}J_i{}^z - \lambda_* J_{ix},$$

from which, taking the skew-symmetric part with respect to indices k and i and making use of (1.5), (1.16) and (3.2)

$$\begin{aligned}&\theta_{kx}A_{ji}{}^{x} - \theta_{ix}A_{jk}{}^{x} + (\nabla_k\lambda_*)f_{ji} - (\nabla_i\lambda_*)f_{jk} \\ =&\frac{1}{4}(c-1)\{J_{i*}(g_{jk} - \xi_j\xi_i) - J_{k*}(g_{ji} - \xi_j\xi_i) \\ &+ w_*(\xi_i f_{kj} - \xi_k f_{ji} - 2\xi_j f_{ki})\}.\end{aligned}$$

Hence it follows that we obtain

$$\begin{aligned}&\theta_{jx}A_{ki}{}^{x} - \theta_{ix}A_{kj}{}^{x} \\ =&(\nabla_i\lambda_*)f_{jk} - (\nabla_j\lambda_*)f_{ik} - 2(\nabla_k\lambda_*)f_{ji} \\ &+ \frac{1}{4}(c-1)\{J_{i*}(g_{jk} - \xi_j\xi_k) - J_{j*}(g_{ik} - \xi_i\xi_k) \\ &+ w_*(\xi_j f_{ik} - \xi_i f_{jk} - 2\xi_k f_{ji})\}.\end{aligned}$$

If we take the skew-symmetric part with respect to j and i in (3.3), and use the last equation, then we have

$$\begin{aligned}&(\nabla_k A_{ir*})f_j{}^r - (\nabla_k A_{jr*})f_i{}^r\\ =&(A_{kj*} - \lambda_* g_{kj})\xi_i - (A_{ki*} - \lambda_* g_{ki})\xi_j + (\nabla_i\lambda_*)f_{jk} - (\nabla_j\lambda_*)f_{ik}\\ &+ (A_{ir*}A_k{}^{rx} - \lambda_* A_{ik}{}^x)J_{jx} - (A_{jr*}A_k{}^{rx} - \lambda_* A_{jk}{}^x)J_{ix}\\ &+ \frac{1}{4}(c-1)\{J_{i*}(g_{jk} - \xi_j\xi_k) - J_{j*}(g_{ik} - \xi_i\xi_k)\\ &+ w_x(\xi_j f_{ik} - \xi_i f_{jk} - 2\xi_k f_{ji}\}.\end{aligned}$$

By transvecting ξ^j and taking account of (1.8), (2.4) and (2.13), we can get

$$\begin{aligned}&(\nabla_k A_{ir*})f_j{}^r\xi^j\\ =&(\alpha-1)(A_{ki*} - \lambda_* g_{ki}) + (f_{ir}\nabla_k\lambda_* - f_{kr}\nabla_i\lambda_* - f_{ik}\nabla_r\lambda_*)\xi^r\\ &+ \frac{1}{4}(c-1)\{(1-\alpha)(J_{i*}\xi_k + w_* f_{ik}) + (f_{kr}\xi_i + 2f_{ir}\xi_k)\xi^r\},\end{aligned}$$

which transvecting g^{ki} and making use of (1.7), (1.16), (2.10) and (3.1),

$$(1-\alpha)\{h^* - (n+1)\lambda_*\} = 0$$

and hence $h^* = (n+1)\lambda_*$. Thus λ^* is constant, which proves the required result.

LEMMA 3.2. *Under the same hypothesis as that in Lemma 3.1, we have $c = 1$.*

Proof. Transforming (3.2) by $J_y{}^i J_z{}^j$ and using (1.6), (2.6) and (2.7), we find

$$\begin{aligned}Q_{yu*}P_z{}^{ux} - Q_{zu*}P_y{}^{ux} =&\frac{1}{4}(c-1)(\delta_y{}^x\delta_{z*} - \delta_z{}^x\delta_{y*} + \delta_z{}^x w_y w_*\\ &- \delta_y{}^x w_z w_* + \delta_{y*}w_z w^x - \delta_{z*}w_y w^x),\end{aligned} \tag{3.4}$$

which enable us to obtain

$$Q_{yz*}P_x{}^{yz} = P^z Q_{xz*} + \frac{1}{4}(c-1)(p-1)(\delta_{x*} - w_* w_x) \tag{3.5}$$

for any index x, where we have defined $P^z = P_x{}^{xz}$ and $p = 2m - n$.

Differentiating (2.6) covariantly and substituting (1.12), we find

$$\begin{aligned}&(\nabla_k A_{jr*})J_y{}^r + A_{kj*}w_y + A_{jr*}A_{ksy}f^{rs}\\ =&(\nabla_k Q_{yz*})J_j{}^z + Q_{yz*}w^z g_{kj} + Q_{yz*}A_{kr}{}^z f_j{}^r.\end{aligned} \tag{3.6}$$

If we take the skew-symmetric part with respect to k and j of (3.6), and make use of (1.6), (1.7), (1.16) and (2.1), then we obtain

$$\begin{aligned}&\frac{1}{4}(c-1)\{w_*(\xi_j J_{ky} - \xi_k J_{jy}) + w_y(\xi_j J_{k*} - \xi_k J_{j*})\}\\ &\quad - \frac{1}{2}(c-1)(\delta_{y*} - w_* w_y)f_{jk} + A_r{}^{s*}A_{ksy}f_j{}^r - A_r{}^{s*}A_{jsy}f_k{}^r\\ =&(\nabla_k Q_{yz*})J_j{}^z - (\nabla_j Q_{yz*})J_k{}^z + Q_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r).\end{aligned}$$

Transvecting f^{jk} to the last equation and using (1.5) $\sim$ (1.7), we can get

$$\begin{aligned}&A_r{}^{s*}A_{ksy}(g^{kr} - \xi^k\xi^r - J^{kz}J_z{}^r)\\ &\quad - \frac{1}{2}(c-1)f_{ji}f^{ji}(\delta_{y*} - w_* w_y) - (c-1)w_* w_y(1-\alpha)\\ =&\xi^k(\nabla_k Q_{yz*})w^z + Q_{yz*}A_{kr}{}^z(g^{kr} - \xi^k\xi^r - J^{ku}J_u{}^r).\end{aligned} \tag{3.7}$$

On the other hand, differentiating (2.14) covariantly and taking account of (1.14) and Lemma 3.1, we find

$$w^z\nabla_k Q_{yz*} - (J_k{}^z + A_{kr}{}^z\xi^r)Q_{yz*} = -\lambda_*(J_{ky} + A_{kry}\xi^r),$$

from which, transvecting ξ^k and using (1.8),

$$\xi^k(\nabla_k Q_{yz*})w^z = (1-\alpha)(Q_{yz*}\lambda^z - \lambda_*\lambda_y). \tag{3.8}$$

Because of (2.4), (2.6), (2.7), (3.1), (3.5) and (3.8), the equation (3.7) is reduced to

$$\begin{aligned}A_{jiy}A^{ji*} =&h^* Q_{y**} + \frac{1}{4}(c-1)(p-1)(\delta_{y*} - w_* w_y)\\ &+ \frac{1}{2}(c-1)\{2(1-\alpha)w_* w_y + f_{ji}f^{ji}(\delta_{y*} - w_* w_y)\},\end{aligned}$$

which together with (2.14) yields

$$w^y A_{jiy} A^{ji*} = h^* \lambda_* w_* + \frac{1}{4}(c-1)(1-\alpha)w_*(4\alpha + p - 1 + 2f_{ji}f^{ji}).$$

By the way, we have from (2.12)

$$w^y A_{jiy} A^{ji*} = h^* \lambda_* w_* - \frac{1}{4}(c-1)(n+3)(1-\alpha)w_*,$$

where we have used (1.7), (1.16), (2.14), (3.1) and Lemma 3.1. Therefore the last two relationships imply that

$$(c-1)w_*(4\alpha + n + p + 2 + 2f_{ji}f^{ji}) = 0$$

and consequently $(c-1)w_* = 0$. Because of (1.14) and (2.4), it follows that we have $(c-1)(J_{j*} + \lambda_* \xi_j) = 0$ and thus $c = 1$ by virtue of (1.6). This completes the proof.

In general we have

$$\frac{1}{2}\Delta(A_{ji*}A^{ji*}) = A^{ji*}\Delta A_{ji*} + \|\nabla A^*\|^2, \tag{3.9}$$

where Δ denotes the operator of the Laplacian.

By Lemma 3.2 the ambient space is a unit sphere. Using various formulas obtained in the sections 2 and 3, (3.9) turns out to be the following:

$$(n+3)\|A^* - \frac{h^*}{n+1}I\|^2 + 2\|\nabla A^*\|^2 = 0,$$

where I denotes the unit tensor (for detail see [3]). Thus we have

THEOREM 3.3. *Let M be an $(n+1)$-dimensional generic submanifold satisfying (2.1) of $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If the function $\alpha(1-\alpha)$ does not vanish almost everywhere, then M is pseudo-umbilical.*

REMARK 1. Hypersurfaces satisfying (2.1) of a Sasakian space form are totally umbilical (cf. [10]).

4. Generic submanifolds tangent to the structure vector

Let M be a generic submanifold satisfying (2.1) of a Sasakian space form $\tilde{M}^{2m+1}(c)$. Suppose that the mean curvature vector of M is nonzero and is parallel in the normal bundle. Moreover, we suppose that the submanifold M is tangent to the structure vector V. Then by (1.4) we have

$$w_x = 0 \tag{4.1}$$

and hence $\xi_t\xi^t = 1$. Thus, (1.6) and (1.7) become respectively

$$J_x{}^t J_t{}^y = \delta_x{}^y, \tag{4.2}$$

$$f_t{}^h J_x{}^t = 0,\ f_{jt}\xi^t = 0. \tag{4.3}$$

We also obtain from (1.14)

$$A_{jrx}\xi^r = -J_{jx}. \tag{4.4}$$

Consequently (2.2) leads to

$$(A_{jr*}J_z{}^r)J_i{}^z - (A_{ir*}J_z{}^r)J_j{}^z + J_{i*}\xi_j - J_{j*}\xi_i = 0.$$

Transvecting this with $J_y{}^i$ and using (4.2) and (4.3), we get

$$A_{jr*}J_y{}^r = P_{yz*}J_j{}^z - \delta_{y*}\xi_j. \tag{4.5}$$

If we transvect (3.2) with $J_y{}^i J_u{}^j$ and use (4.4) and (4.5), then we obtain

$$P_{yz*}P_{xu}{}^z - P_{zu*}P_{xy}{}^z = \frac{1}{4}(c+3)(\delta_{u*}\delta_{xy} - \delta_{y*}\delta_{xu}) \tag{4.6}$$

and thus

$$P_{xz*}P^{xz}{}_y - P^z P_{zy*} = \frac{1}{4}(c+3)(p-1)\delta_{y*}. \tag{4.7}$$

For the shape operator A^* in the direction of the mean curvature vector a function $h_{(m)}$ for any integer $m \geq 2$ is introduced as follows:

$$h_{(m)} = \mathrm{Tr}\ (A^*)^m.$$

LEMMA 4.1. *Let M be a generic submanifold satisfying (2.1) of $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If M is tangent to the structure vector, then we have*

$$A_{jiy}A^{ji*} = h^*P_{y**} + \frac{1}{4}(n-1)(c+3)\delta_{y*} + 2\delta_{y*}, \tag{4.8}$$

$$h_{(3)} = h^*\|P_{z**}\|^2 + \frac{1}{4}(n-2)(c+3)P_{***} + 3P_{***} + \frac{1}{4}(c+3)h^*, \tag{4.9}$$

where $\|P_{z**}\|^2 = P_{z**}P^{z**}$.

Proof. Differentiating (4.5) covariantly along M and making use of (1.11), (1.13), (2.1) and (4.1), we find

$$(\nabla_k A_{jr*})J_y{}^r + A_r{}^s{}_*A_{ksy}f_j{}^r = (\nabla_k P_{yz*})J_j{}^z + P_{yz*}A_{kr}{}^z f_j{}^r - \delta_{y*}f_{kj}.$$

Hence, if we take the skew-symmetric part with respect to k and j, then we obtain

$$\begin{aligned} &A_r{}^s{}_*A_{ksy}f_j{}^r - A_r{}^s{}_*A_{jsy}f_k{}^r \\ =&(\nabla_k P_{yz*})J_j{}^z - (\nabla_j P_{yz*})J_k{}^z + P_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r) \\ &+ \frac{1}{2}(c+3)\delta_{y*}f_{jk}, \end{aligned} \tag{4.10}$$

where we have used (1.16) and (4.1) $\sim$ (4.3).

Transvecting (4.10) with f^{jk} and using (1.5) and (4.3), we can get

$$\begin{aligned} &A_r{}^s{}_*A_{ksy}(g^{kr} - \xi^k\xi^r - J^{kw}J_w{}^r) \\ =&P_{yz*}A_{kr}{}^z(g^{kr} - \xi^k\xi^r - J^{kw}J_w{}^r) + \frac{1}{4}(c+3)(n-p)\delta_{y*}, \end{aligned}$$

which joined with (3.1), (4.4) and (4.5) yields

$$\begin{aligned} &A_{jiy}A^{ji*} + J_*{}^sA_{ksy}\xi^k - (P_{wz*}J^{sw} - \delta_{z*}\xi^s)A_{ksy}J^{kz} \\ =&h^*P_{y**} - P^zP_{zy*} + \frac{1}{4}(c+3)(n-p)\delta_{y*}. \end{aligned}$$

Hence we have

$$A_{jiy}A^{ji*} = h^*P_{y**} + P_{wz*}P^{wz}{}_y - P^zP_{yz*} + 2\delta_{y*} + \frac{1}{4}(c+3)(n-p)\delta_{y*},$$

where we have used (1.12) and (4.4), which together with (4.2) and (4.7) implies that (4.8) is valid.

When $y = *$ in (4.10), we have

$$\begin{aligned} -2A_{js}{}^*A_r{}^{s*}f_k{}^r =&(\nabla_kP_{z**})J_j{}^z - (\nabla_jP_{z**})J_k{}^z \\ &+ P_{z**}(A_{kr}{}^zf_j{}^r - A_{jr}{}^zf_k{}^r) + \frac{1}{2}(c+3)f_{jk} \end{aligned} \tag{4.11}$$

because of (2.1).

By the way we have

$$A^{jr*}f_r{}^t(\nabla_tP_{z**})J_j{}^z = 0$$

with the aid of (4.3) and (4.5). Hence, by transvecting $A_s{}^{j*}f^{ks}$ to (4.11) and using (1.5), we find

$$\begin{aligned} &A_r{}^{t*}A_{ts}{}^*A_i{}^{s*}(g^{ri} - \xi^r\xi^i - J^{rz}J_z{}^i) \\ =&P_{z**}A_{rs}{}^zA_i{}^{r*}(g^{si} - \xi^s\xi^i - J^{su}J_u{}^i) \\ &+ \frac{1}{4}(c+3)A_{ri}{}^*(g^{ri} - \xi^r\xi^i - J^{rz}J_z{}^i), \end{aligned}$$

or, using (4.2) $\sim$ (4.5)

$$\begin{aligned} &h_{(3)} - A^{st*}(P_{yz*}J_t{}^y - \delta_{z*}\xi_t)(P_x{}^{z*}J_s{}^x - \delta^{z*}\xi_s) \\ =&P_{z**}A_{ji}{}^zA^{ji*} - P_{z**}A_{js}{}^zJ^{sw}(P_{yw*}J^{jy} - \delta_{w*}\xi^j) \\ &+ \frac{1}{4}(c+3)(h^* - P^*). \end{aligned}$$

Thus it follows that we obtain

$$\begin{aligned} &h_{(3)} - P_{yz*}P_x{}^{z*}P^{xy*} \\ =&h^*\|P_{z**}\|^2 + \frac{1}{4}(c+3)(n-1)P_{***} - P_{z**}P_{yw*}P^{zwy} + 3P_{***} \\ &+ \frac{1}{4}(c+3)(h^* - P^*) \end{aligned}$$

because of (4.2), (4.4) and (4.8). Since we have from (4.6)

$$P_{yz*}P_x{}^{z*}P^{xy*} - P_{z**}P_{xy*}P^{xyz} = \frac{1}{4}(c+3)(P^* - P_{***}),$$

above equation can be written as

$$h_{(3)} = h^*\|P_{z**}\|^2 + \frac{1}{4}(c+3)(n-2)P_{***} + 3P_{***} + \frac{1}{4}(c+3)h^*,$$

which proves (4.9). Hence, Lemma 4.1 is proved.

LEMMA 4.2. *Under the same hypothesis as that in Lemma 4.1, the function $h_{(2)}$ is harmonic.*

Proof. By definition, we have $P_{***} = A_{ji}{}^*J_*{}^jJ_*{}^i$. Differentiating this covariantly and making use of (1.11), (4.1) and (4.5), we find

$$\nabla_k P_{***} = (\nabla_k A_{ji}{}^*)J_*{}^jJ_*{}^i + 2(P_{z**}J_j{}^z - \xi_j)A_{kr}{}^*f^{jr},$$

which gives $\nabla_k P_{***} = (\nabla_k A_{ji}{}^*)J_*{}^jJ_*{}^i$ because of (4.3). Thus the Laplacian of the function P_{***} is given by

$$\Delta P_{***} = (\Delta A_{ji}{}^*)J_*{}^jJ_*{}^i + 2(\nabla_k A_{ji}{}^*)J_*{}^iA_r{}^{k*}f^{jr},$$

which together with (1.16), (2.1) and (4.1) yields

$$\Delta P_{***} = (\Delta A_{ji}{}^*)J_*{}^jJ_*{}^i - \frac{1}{2}(c-1)(h^* - P^*). \tag{4.12}$$

On the other hand, since the submanifold M has parallel mean curvature vector field, the Laplacian $\Delta A_{ji}{}^*$ of A^* is given , using the Ricci formula for A^* and (1.16) and (4.1), by

$$\begin{aligned}\Delta A_{ji}{}^* =& S_{jr}A_i{}^{r*} - R_{kjih}A^{kh*} \\ &+ \frac{1}{4}(c-1)\nabla_k(J_*{}^kf_{ji} + J_{j*}f_i{}^k + 2J_{i*}f_j{}^k).\end{aligned} \tag{4.13}$$

Thus, it follows that we get

$$(\Delta A_{ji}{}^{*})J_{*}{}^{j}J_{*}{}^{i} = S_{jr}A_{i}{}^{r*}J_{*}{}^{j}J_{*}{}^{i} - R_{kjih}A^{kh*}J_{*}{}^{j}J_{*}{}^{i} + \frac{3}{4}(c-1)(h^{*}-P^{*})$$

because of (1.5), (1.10), (1.11), (4.5) and (4.6). Therefore (4.12) turns out to be

$$\Delta P_{***} = S_{jr}A_{i}{}^{r*}J_{*}{}^{j}J_{*}{}^{i} - R_{kjih}A^{kh*}J_{*}{}^{j}J_{*}{}^{i} + \frac{1}{4}(c-1)(h^{*}-P^{*}). \quad (4.14)$$

By the way, from (1.18) we obtain

$$\begin{aligned} &S_{ji}J_{*}{}^{j}A_{r}{}^{i*}J_{*}{}^{r} \qquad (4.15)\\ =&h^{*} - P^{*} + \frac{1}{4}(n-1)(c+3)P_{***} + h^{*}\|P_{z**}\|^{2} - P_{z**}P_{ux*}P^{zux} \end{aligned}$$

because of (3.1) and (4.1) $\sim$ (4.5). We also have by (1.15)

$$\begin{aligned} R_{kjih}J_{*}{}^{j}J_{*}{}^{i}A^{kh*} =& \frac{1}{4}(c+3)(h^{*}-P_{***}) + h^{*}\|P_{z**}\|^{2} \qquad (4.16)\\ &+ \frac{1}{4}(n-1)(c+3)P_{***} - P_{ux*}P_{z}{}^{u*}P^{zx*}, \end{aligned}$$

where we have used (4.3) $\sim$ (4.5) and (4.8).

Substituting (4.15) and (4.16) into (4.14) and taking account of (4.8), we get $\Delta P_{***} = 0$. Since we have from (4.8)

$$h_{(2)} = h^{*}P_{***} + \frac{1}{4}(c+3)(n-1) + 2, \quad (4.17)$$

it is clear that $\Delta h_{(2)} = 0$ because the mean curvature vector field is parallel. Therefore we arrive at the conclusion.

By a straightforward computation, we have

$$A_{j}{}^{rx}A_{irx}A^{is*}A_{s}{}^{j*} = A_{jr}{}^{x}A_{isx}A^{rs*}A^{ji*} + \{\frac{1}{4}(c-1)\}^{2}(p-1), \quad (4.18)$$

where we have used (3.2), (4.2), (4.4) and (4.7).

We also have

$$A^{kh*}A^{ji*}f_{kj}f_{hi} = h_{(2)} - P^z P_{z**} - \frac{1}{4}(c-1)(p-1) - (p+1) \quad (4.19)$$

because of (1.5), (2.1), (4.3), (4.4), (4.5) and (4.7).

Making use of (1.15) and (1.18), we can verify the following:

$$S_{js}A_i{}^{s*}A^{ji*} - R_{kjih}A^{kh*}A^{ji*} = \frac{1}{16}(c-1)^2(n-p), \quad (4.20)$$

where we have used (4.4), (4.5), (4.7), (4.8), (4.9), (4.17), (4.18) and (4.19).

By (1.10) we have

$$\nabla_k f_i{}^k = n\xi_i - h^* J_{i*} - A_{ir}{}^x J_x{}^r$$

because of (3.1). Hence we have

$$A^{ji*}\nabla_k(J_{j*}f_i{}^k) = A^{ji*}A^*_{kr}f_j{}^r f_i{}^k + (P_{z**}J^{iz} - \xi^i)(n\xi_i - h^* J_{i*} + A_i{}^{rx}J_{xr})$$

by virtue of (1.11) and (4.5), or using (4.2), (4.4), (4.17) and (4.19) we obtain

$$A^{ji*}\nabla_k(J_{j*}f_i{}^k) = \frac{1}{4}(c-1)(n-p) \quad (4.21)$$

Multiplying A^{ji*} to (4.13) and summing for j and i, and making use of (4.20) and (4.21), we have

$$A^{ji*}\Delta A^*_{ji} = -\frac{1}{8}(c-1)^2(n-p). \quad (4.22)$$

Thus by Lemma 4.2 and (3.9), we obtain

$$\|\nabla_k A_{ji}{}^*\|^2 = \frac{1}{8}(c-1)^2(n-p). \quad (4.23)$$

On the other hand, we easily verify that

$$\|\nabla_k A_{ji}{}^* + \frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{i*}f_{kj})\|^2 = \|\nabla_k A_{ji}{}^*\|^2 - \frac{1}{8}(c-1)^2(n-p),$$

where we have used (1.5), (1.16), (4.2) and (4.3). Thus, we have

$$\nabla_k A_{ji}{}^{*} = -\frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{i*}f_{kj}). \tag{4.24}$$

For any point q in M we can choose a local orthonormal frame field $\{E_i\}$ so that the shape operator A^* in the direction of the mean curvature vector field is diagonalizable at that point q, say $A_{ji}{}^{*} = \lambda_j \delta_{ji}$.

Transvecting (4.24) with $(A^{ji*})^{a-1}$ for any integer $a \geq 2$ and taking account of (4.3) $\sim$ (4.5), we obtain $\nabla_k h_{(a)} = 0$ and hence $h_{(a)}$ is constant. Since we have

$$h_{(a)} = \sum_i (\lambda_i)^a, \quad (a = 1, 2, \cdots),$$

it is seen that λ_i is constant. Thus we obtain

LEMMA 4.3. *Under the same assumptions as that in Lemma 4.1, each eigenvalue of A^* is constant.*

We denote by σ_{ji} the sectional curvature of M spanned by E_j and E_i. Then by (4.20) we have

$$\sum_{j,i} (\lambda_i - \lambda_j)^2 \sigma_{ji} = \frac{1}{8}(c-1)^2(n-p) \geq 0$$

because of (4.23). Thus, if $\sigma_{ji} \leq 0$, then $(c-1)^2(n-p) = 0$, and consequently $\nabla A^* = 0$ by (4.23). Moreover, we have $c = 1$ or $n = p$. If $n = p$, then $f = 0$ and M is a totally real submanifold of $\tilde{M}^{2m+1}(c)$ tangent to the structure vector field V (cf. [6]). Thus we have

THEOREM 4.4. *Let M be an $(n+1)$-dimensional generic submanifold tangent to the structure vector field of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector field. If $A^* f = f A^*$, and if the sectional curvature of M is nonpositive , then the shape operator A^* in the direction of the mean curvature vector field is parallel. Moreover, $c = 1$, or M is totally real in $\tilde{M}^{2m+1}(c)$ with respect to ϕ.*

REMARK 2.. An example satisfying the conditions of Theorem 4.4 is shown in [2].

5. Main theorems

Let M be a generic submanifold of a Sasakian space form. Suppose that the mean curvature vector of M is nonzero and is parallel in the normal bundle, and that $A^*f = fA^*$ holds on M.

Let $M_1 = \{p \in M | \alpha(p) = 0\}$, $M_2 = \{p \in M | \alpha(p) = 1\}$ and $\bar{M} = M - M_1 \cup M_2$. Then we have $M = \bar{M} \cup M_1 \cup M_2$. Because of (1.4) the sets M_1 and M_2 are then geometrically characterized as follows: The structure vector field V in the ambient space is tangent to the generic submanifold M at any point in the set M_1, and the vector field V is normal to M at each point in M_2.

If we suppose that there is an open set O contained in M_2, then we obtain $f_{ji} + A_{jix}w^x = 0$ in O by (1.13) and hence $A_{jix}w^x = 0$ because f_{ji} is skew-symmetric and $A_{jix}w^x$ is symmetric. Accordingly we have $h^*w_* = 0$ on O by (3.1) and thus $w_* = 0$ since the mean curvature vector is not zero. Therefore, it is, using (1.14), seen that $J_{j*} = 0$ on O. This contradicts (1.6). Thus the set M_2 is bordered. Hence we may discuss properties of eigenvalues of A^* and the covariant derivative ∇A^* only on $\bar{M} \cup M_1$ since it is continuous. Suppose that there exists a connected component $C(M_1)$ of M_1. Then, by Lemma 4.3 eigenvalues of A^* are constant on $C(M_1)$.

When $\bar{M}$ is not empty, as a consequence of Theorem 3.3, we see that each eigenvalue λ_i of A^* is constant. Therefore we obtain λ_i = const. on each component of $\bar{M}$ and M_1. Since λ_i is continuous, it follows that each eigenvalue λ_i=const. When $\bar{M}$ is empty, Lemma 4.3 implies λ_i = const. Thus we have

THEROEM 5.1. *Let M be a generic submanifold of a Sasakian space form with nonvanishing parallel mean curvature vector. If the shape operator A^* in the direction of the mean curvature vector commutes with the structure tensor f induced on M, then each eigenvalue of A^* is constant.*

By the same argument used in the proof of Theorem 5.1, we can, taking account of Theorem 3.3 and Theorem 4.4, verify the following:

THEOREM 5.2. *Let M be a generic submanifold of a Sasakian space form with nonvanishing parallel mean curvature vector field. If $A^*f = fA^*$, and if the sectional curvature of M is nonpositive, then the shape operator A^* in the direction of the mean curvature vector field is parallel.*

From Theorem 3.3 and (4.24) we have

COROLLARY 5.3 ([3]). *Let M be a generic submanifold of an odd-dimensional sphere with nonvanishing parallel mean curvature vector field. If $A^*f = fA^*$, then A^* is parallel.*

We will consider the case where A^* is parallel on M. Let $\mu_1, \cdots, \mu_s$ be mutually distinct eigenvalues of A^* and $n_1, \cdots, n_s$ their multiplicities. Since A^* is parallel, the smooth distribution $T_a (a = 1, \cdots, s)$ which consists of all eigenvectors accociated with the eigenvalues can be defined and is parallel. M is assumed to be simply connected and complete, then by means of the de Rham decomposition theorem, the submanifold is a product of Riemannian manifolds $M_1 \times \cdots \times M_s$, where the tangent bundle of M_a correspond to T_a. Since the shape operator A^* restricted to T_a is propotional to the identity transformation of T_a and each submanifold M_a is totally geodesic in M, the mean curvature vector of M is an umbilical section of M_a in $\tilde{M}^{2m+1}(c)$. Thus, by means of the above arguments and Theorem 5.2 we have

THEOREM 5.4. *Let M be a complete and simply connected generic submanifold of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector. If the sectional curvature of M is nonpositive and if the shape operator A^* in the direction of the mean curvature vector commutes with the structure tensor f induced on M, then M is a product of Riemannian manifolds $M_1 \times \cdots \times M_s$, where s is the number of distinct eigenvalues of A^*, and the mean curvature vector of M is an umbilical section of $M_a (a = 1, \cdots, s)$.*

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