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DEPTHS OF THE REES ALGEBRAS AND THE ASSOCIATED GRADED RINGS

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring (R, m), we mean a Noetherian ring R which has a unique maximal ideal m. By $\dim(R)$ we always mean the Krull dimension of R. Let I be an ideal in a ring R and t an indeterminate over R. Then the Rees algebra $R[It]$ and the associated graded ring $gr_I(R)$ of I are defined to be

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$$

and

$$gr_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

These rings are important not only algebraically, but geometrically as well. For example, $\mathrm{Proj}(R[It])$ is the blow-up of $\mathrm{Spec}(R)$ with respect to I.

The purpose of this paper is to investigate the relationship between the depths of the Rees algebra $R[It]$ and the associated graded ring $gr_I(R)$ of an ideal I in a local ring (R, m) of $\dim(R) > 0$. The relationship between the Cohen-Macaulayness of these two rings has been studied extensively. Let (R, m) be a local ring and I an ideal of R. An ideal J contained in I is called a reduction of I if $JI^n = I^{n+1}$ for some integer

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$n \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I. The reduction number of I with respect to J is defined by

$$r_J(I) = \min\{n \geq 0 \mid JI^n = I^{n+1}\}.$$

The reduction number of I is defined by

$$r(I) = \min\{r_J(I) \mid J \quad \text{is a minimal reduction of } I\}.$$

S. Goto and Y. Shimoda characterized the Cohen-Macaulay property of the Rees algebra of the maximal ideal of a Cohen-Macaulay local ring in terms of the Cohen-Macaulay property of the associated graded ring of that maximal ideal and the reduction number of that maximal ideal. Let us state their theorem.

THEOREM 1.1. *([4], Theorem 3.1) Let (R, m) be a Cohen-Macaulay local ring of dimension $d > 0$ and assume that R/m is infinite. Then the following conditions are equivalent.*
(1) $R[mt]$ is a Cohen-Macaulay ring.
(2) $gr_m(R)$ is a Cohen-Macaulay ring and $r(m) \leq d-1$.

In a number of cases, this theorem gives a test for determining whether or not $R[mt]$ is Cohen-Macaulay, because $r(m)$ is reasonable to compute. For example, let $R = k[[X^2, X^3]]$ and $m = (X^2, X^3)R$, where k is a field and X is variable over k. Then R is one-dimensional local domain and $r(m) = 1$. Hence $R[mt]$ is not Cohen-Macaulay by Theorem 1.1. More generally, if (R, m) is any one-dimensional local domain which is not a rank one discrete valuation domain, then $R[mt]$ is not Cohen-Macaulay by Theorem 1.1.

Let (R, m) be a local ring and I an ideal of R. The analytic spread of I, denoted by $l(I)$, is defined to be $\dim(R[It]/mR[It])$. In [13], it is shown that $ht(I) \leq l(I) \leq \dim(R)$. An ideal I is called equimultiple if $l(I) = ht(I)$. If R/m is an infinite field, then $l(I)$ is the least number of elements generating a reduction of I ([13]). In particular, all m-primary ideals are equimultiple. U. Grothe, M. Herrmann and U. Orbanz generalized Theorem 1.1 to the case of all "equimultiple ideals". We now state the result of Grothe - Herrmann - Orbanz.

THEOREM 1.2. *([5], Theorem 4.8) Let (R, m) be a Cohen-Macaulay local ring having an infinite residue field and I an equimultiple ideal of height s. Assume that $s > 0$. Then the following conditions are equivalent.*
(1) $R[It]$ is a Cohen-Macaulay ring.
(2) $gr_I(R)$ is a Cohen-Macaulay ring and $r(I) \leq s - 1$.

In general, it is known (*cf.* [9], Proposition 1.1) that if R and $R[It]$ are Cohen-Macaulay, then $\text{depth}(R[It]) = \text{depth}(gr_I(R)) + 1$. On the other hand, if $gr_I(R)$ is Cohen-Macaulay, then $\text{depth}(R[It]) \leq 1 + \text{depth}(gr_I(R))$ (see Lemma 3.1). We shall prove that the following equality

$$\text{depth}(R[It]) = \text{depth}(gr_I(R)) + 1$$

always holds for ideal I under negation of the Cohen-Macaulay assumption on $gr_I(R)$ and the condition that R is normally Cohen-Macaulay along I. We also characterize that the property of Cohen-Macaulayness of $R[It]$ and $gr_I(R)$ are equivalent for an equimultiple ideal I by imposing the condition of a regular local ring on R. As a general reference, we refer the reader to [11] for any unexplained notation and terminology.

2. Preliminaries

Let R be a Noetherian ring and I an ideal of R. Given an element $a \in R$, we define

$$v_I(a) = \begin{cases} n & \text{if } a \in I^n \backslash I^{n+1} \\ \infty & \text{if } a \in \cap_{n \geq 1} I^n. \end{cases}$$

When $v_I(a) = n \neq \infty$, the residue class of a in I^n/I^{n+1} is called the leading form of a and denoted by a^*. If $v_I(a) = \infty$, then we set $a^* = 0$.

LEMMA 2.1. *Let R be a Noetherian ring and I an ideal in R. Let n be a non-negative integer and $b \in R$. Assume that $bR \cap I^i = bI^{i-n}$ for $i \geq n$. Let $R_1 = R/bR$ and $I_1 = IR_1$. Then*

$$R_1[I_1t] \cong \frac{R[It]}{(b, bt, \cdots, bt^n)}$$

as graded R-algebras.

Proof. : Note that $bR \cap I^j = bR$ for $j < n$. Let $\phi : R[It] \longrightarrow R_1[I_1t]$ denote the canonical epimorphism. Put $K = \mathrm{Ker}\phi$. Then K is a homogeneous ideal in $R[It]$.

Claim : $K = (b, bt, \cdots, bt^n)$.

$\supseteq$: It is obvious.

$\subseteq$: Let z be a homogeneous element of K with $\deg z = l \geq 0$. Write $z = \alpha t^l$ with $\alpha \in I^l$. Then we have $\alpha \in bR \cap I^l$. We have two cases : (1) when $l \geq n$, and (2) when $l < n$.

Case (1) : $l \geq n$. By assumption we write $\alpha = bc$ with $c \in I^{l-n}$, and hence

$$z = \alpha t^l = bct^l = bt^n \cdot ct^{l-n} \in (bt^n)R[It].$$

Case (2) : $l < n$. From the note, we write $\alpha = br$ with $r \in R$, and hence

$$z = \alpha t^l = rbt^l \in (bt^l)R[It]. \qquad \blacksquare$$

LEMMA 2.2. *Let R be a Noetherian ring, I an ideal in R and $a \in R$. Assume that a is a non-zero-divisior on R and $aR \cap I^n = aI^{n-1}$ for $n \geq 1$. Then*

(1) $(aR[It] : at) = IR[It]$.

(2) There exists an exact sequence

$$0 \longrightarrow gr_I(R) \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \longrightarrow 0$$

of graded $R[It]$-modules.

Proof. : (1) $\supseteq$: Let $f \in IR[It]$. Write $f = f_0 + f_1t + \cdots + f_st^s$, where $f_i \in I^{i+1}$, $i = 0, 1, \cdots, s$. Then we have

$$f \cdot at = a(f_0t + f_1t^2 + \cdots + f_st^{s+1}) \in aR[It].$$

$\subseteq$: Let $f \in (aR[It] : at)$ with $f = f_0 + f_1t + \cdots + f_lt^l \in R[It]$. Then $f \cdot at = ag$, where $g = g_0 + g_1t + \cdots + g_{l+1}t^{l+1} \in R[It]$. Hence we have

$$ag_0 + (ag_1 - af_0)t + \cdots + (ag_{l+1} - af_l)t^{l+1} = 0.$$

By the nature of a, $f_i = g_{i+1} \in I^{i+1}$ for $i = 0, 1, \cdots, l$, which concludes the proof of (1).
(2) Consider the exact sequence

$$0 \longrightarrow \frac{(a, at)R[It]}{(a)R[It]} \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \frac{R[It]}{(a, at)R[It]} \longrightarrow 0$$

of graded $R[It]$-modules. Moreover

$$\begin{aligned} \frac{(a, at)R[It]}{aR[It]} &\cong \frac{(at)R[It]}{aR[It] \cap (at)R[It]} = \frac{(at)R[It]}{(aR[It] : at)(at)} \\ &\cong \frac{R[It]}{(aR[It] : at)} = \frac{R[It]}{IR[It]} \qquad \text{by (1)} \\ &\cong gr_I(R), \end{aligned}$$

and

$$\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \cong \frac{R[It]}{(a, at)R[It]} \qquad \text{by Lemma 2.1} \qquad \blacksquare$$

Notation : Let $G = \oplus_{n \geq 0} G_n$ be a non-negatively graded Noetherian ring such that G_0 is a local ring and A a finitely generated graded G-module. Then we define depth(A) to be depth$_{G_N}(A_N)$, where N is the unique homogeneous maximal ideal of G. We let G^+ denote the ideal $\oplus_{n \geq 1} G_n$.

LEMMA 2.3. *(cf. [3], Lemma 1.1) Let G be a non-negatively graded Noetherian ring such that G_0 is a local ring and A, B and C be finitely generated graded G-modules. Suppose there is an exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where the maps are all homogeneous. Then either
(1) $depth A \geq depth B = depth C$, or
(2) $depth B \geq depth A = depth C + 1$, or
(3) $depth C > depth A = depth B$.

Proof. : The proof follows from the Ext characterization of depth, and the long exact sequence for Ext. $\blacksquare$

DEFINITION 2.4. Let (R, m) be a local ring and I an ideal of R. We say R is normally Cohen-Macaulay along I if

$$\text{depth}\,(I^n/I^{n+1}) = \dim\,(R/I) \qquad \text{for all } n \geq 0.$$

REMARKS. : (1) Let (R, m) be a local ring. Then R is normally Cohen-Macaulay along any m-primary ideal I.
(2) Let (R, m) be a quasi-unmixed local ring and I an ideal in R with $ht(I) > 0$. Assume that R is normally Cohen-Macaulay along I. Then I is an equimultiple ideal.
(3) Let (R, m) be a local ring and I an ideal of R, and suppose that R is normally Cohen-Macaulay along I. Suppose that b^*, the image of b in R/I, is a $gr_I(R)$-regular element. Then R/bR is normally Cohen-Macaulay along $I(R/bR)$.

Proof. : (1) It is trivial.
(2) Recall that $\dim(R) = \dim(R/I) + ht(I)$ since R is a quasi-unmixed local ring. R/I^n is Cohen-Macaulay for all $n \geq 1$ ([6], Lemma 3.8). Then we have by a result of L. Burch ([1], Corollary in pp. 373)

$$\begin{aligned} l(I) &\leq \dim(R) - \min_n\{\text{depth}(R/I^n)\} \\ &= \dim(R) - \text{depth}(R/I^{n_0}), \qquad \text{for some integer } n_0 \\ &= \dim(R) - \dim(R/I^{n_0}) \\ &= ht(I^{n_0}) \\ &= ht(I). \end{aligned}$$

(3) Put $R_1 = R/bR$ and $I_1 = IR_1$. We have the following isomorphisms

$$(I_1)^n/(I_1)^{n+1} \cong \frac{I^n + bR}{I^{n+1} + bR} \cong \frac{I^n}{I^{n+1} + bI^n} \cong \frac{I^n/I^{n+1}}{b(I^n/I^{n+1})}.$$

Since b^* is a $gr_I(R)$-regular element, b is a non-zero-divisor on I^n/I^{n+1} for all $n \geq 0$. Hence, we have

$$\begin{aligned} \text{depth}\,(I_1^n/I_1^{n+1}) &= \text{depth}\,(I^n/I^{n+1}) - 1 \\ &= \dim(R/I) - 1 \\ &= \dim(R_1/I_1). \qquad \blacksquare \end{aligned}$$

3. Depths of the Rees algebras and the associated graded rings

LEMMA 3.1. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and I an ideal of $ht(I) > 0$. Then*

$$depth(R[It]) \leq depth(gr_I(R)) + 1.$$

Proof. : Consider the exact sequences

$$0 \longrightarrow ItR[It] \longrightarrow R[It] \longrightarrow R \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow IR[It] \longrightarrow R[It] \longrightarrow gr_I(R) \longrightarrow 0 \tag{2}$$

of $R[It]$-modules. From (2) we have that by Lemma 2.3, either

$$\text{depth}(R[It]) \geq \text{depth}(IR[It]) = \text{depth}(gr_I(R)) + 1,$$

or

$$\text{depth}(gr_I(R)) \geq \text{depth}(R[It]).$$

In the second case we are done. Hence we assume that

$$\text{depth}(IR[It]) = \text{depth}(gr_I(R)) + 1. \tag{3}$$

From (1) it follows that by Lemma 2.3, either

$$\text{depth}(ItR[It]) \geq \text{depth}(R[It]),$$

or

$$\text{depth}(R[It]) \geq \text{depth}(ItR[It]) = \text{depth}(R) + 1.$$

But since $IR[It] \cong ItR[It]$ as $R[It]$-modules, we have

$$\text{depth}(IR[It]) = \text{depth}(ItR[It]). \tag{4}$$

First, if $\mathrm{depth}(ItR[It]) \geq \mathrm{depth}(R[It])$, then

$$\begin{aligned}\mathrm{depth}(gr_I(R)) + 1 &= \mathrm{depth}(IR[It]) && \text{by } (3)\\ &= \mathrm{depth}(ItR[It]) && \text{by } (4)\\ &\geq \mathrm{depth}(R[It]).\end{aligned}$$

Second, if $\mathrm{depth}(ItR[It]) = \mathrm{depth}(R) + 1$, then

$$\begin{aligned}\mathrm{depth}(gr_I(R)) + 1 &= \mathrm{depth}(IR[It]) && \text{by } (3)\\ &= \mathrm{depth}(ItR[It]) && \text{by } (4)\\ &= \mathrm{depth}(R) + 1\\ &= \dim(R) + 1 && (\mathrm{R} : \mathrm{CML})\\ &= \dim(R[It])\\ &\geq \mathrm{depth}(R[It]).\end{aligned}$$

Thus, in all cases we have

$$\mathrm{depth}(R[It]) \leq \mathrm{depth}(gr_I(R)) + 1. \qquad \blacksquare$$

LEMMA 3.2. *Let V be a finite-dimensional vector space over the infinite field K, and let $H_1, \cdots, H_n$ be proper subspaces of V. Then there exists $v \in V$ such that $v \notin H_1 \cup \cdots \cup H_n$.*

Proof. : We proceed by induction on n. If $n = 1$, then it is clear. If $n > 1$, then we can choose an element $\alpha \in V$ such that $\alpha \notin H_1 \cup \cdots \cup H_{n-1}$ by inductive hypothesis. By the nature of H_n, there exists an element $\beta \in V \backslash H_n$. Suppose that $H_1 \cup \cdots \cup H_n = V$. Since K is infinite, there are distinct elements $r_1, \cdots, r_{n+1}$ in K such that $\alpha + r_1\beta, \cdots, \alpha + r_{n+1}\beta$ are in V. By the pigeonhole principle, two of them must be in the same subspace, say $\alpha + r_i\beta, \alpha + r_j\beta$ are in H_k for some k, where $i \neq j$. If $k = n$, then $(\alpha + r_i\beta) - (\alpha + r_j\beta) = (r_i - r_j)\beta \in H_n$. Hence $\beta \in H_n$, which is a contradiction to the choice of β. If $k < n$, then $(r_i - r_j)\beta \in H_k$, and hence $\beta \in H_k$. Since $\alpha + r_i\beta \in H_k$, it follows that $\alpha \in H_k$, which is a contradiction to the choice of α. $\blacksquare$

LEMMA 3.3. *Let (R, m) be a local ring and I an ideal in R of $ht(I) > 0$. Suppose that*

$$depth(I^n/I^{n+1}) > 0 \qquad \text{for all} \quad n \geq 0.$$

Then we can find an element $x \in m$ which is a non-zero-divisor on R/I^n for all $n \geq 0$.

Proof. : Since $\bigcup_n \mathrm{Ass}_{R/I}(I^n/I^{n+1}) \subseteq \mathrm{Ass}_{R/I}(gr_I(R))$ and $\mathrm{Ass}_{R/I}(gr_I(R))$ is a finite set (*cf*, [12], Proposition 1.3), and hence $\bigcup_n \mathrm{Ass}_{R/I}(I^n/I^{n+1})$ is a finite set. We can choose an element $x \in m$ which is a non-zero-divisor on I^n/I^{n+1} for all $n \geq 0$.

Claim : x is a non-zero-divisor on R/I^{n+1} for all $n \geq 0$.

This will be done by induction on n. The assertion is clear for $n = 0$. So we assume $n \geq 1$. Since x is a non-zero-divisor on I^n/I^{n+1} and on R/I^n, x is a non-zero-divisor on R/I^{n+1} by considering a short exact sequence. ∎

THEOREM 3.4. *Let (R, m) be a positive integer d-dimensional Cohen-Macaulay local ring having an infinite residue field k and I an ideal with $ht(I) > 0$. Assume that $gr_I(R)$ is not Cohen-Macaulay and R is normally Cohen-Macaulay along I. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

Proof. : The inequality $\leq$ holds by Lemma 3.1. We now prove the other inequality. We proceed by induction on $r = \dim(R/I)$. We have two cases : (1) when $r = 0$, and (2) when $r > 0$.

Case (1) : $r = 0$. In this case I is an m-primary ideal of R. We now proceed by induction on $d = \dim(R)$. Since the inequality is trivial if either $d = 1$ or $\mathrm{depth}(gr_I(R)) = 0$, we may assume that $d \geq 2$ and $\mathrm{depth}(gr_I(R)) \geq 1$. Since I is an m-primary ideal of R, any homogeneous element of degree 0 that is not a unit is nilpotent in $gr_I(R)$. Hence there exists a regular element in $gr_I(R)^+$. That is, $gr_I(R)^+ \not\subseteq \bigcup\{Q \mid Q \in \mathrm{Ass}(gr_I(R))\}$. For each $Q \in \mathrm{Ass}(gr_I(R))$, $((Q \cap I/I^2) + mI/I^2)/(mI/I^2)$ is a proper k-vector subspace of I/mI by Nakayama's Lemma. Since k is infinite, we can choose $a \in I \backslash mI$

such that the image of a in I/I^2, a^*, is not in any associated prime Q of $gr_I(R)$ by Lemma 3.2. That is, a^* is a $gr_I(R)$-regular element. Hence a is a non-zero-divisor on R and $aR \cap I^n = aI^{n-1}$ for all $n \geq 1$ (*cf* : [14], Corollary 2.7). We have an exact sequence

$$0 \longrightarrow gr_I(R) \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \longrightarrow 0$$

of $R[It]$-modules by Lemma 2.2. Applying Lemma 2.3, we see that either

$$\operatorname{depth}(gr_I(R)) \geq \operatorname{depth}\left(\frac{R[It]}{(a)}\right) = \operatorname{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right),$$

or

$$\operatorname{depth}\left(\frac{R[It]}{(a)}\right) \geq \operatorname{depth}(gr_I(R)) = \operatorname{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right) + 1,$$

or

$$\operatorname{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right) > \operatorname{depth}(gr_I(R)) = \operatorname{depth}\left(\frac{R[It]}{(a)}\right).$$

But as a^* is a $gr_I(R)$-regular element, $gr_I(R)/(a^*) \cong gr_{I_1}(R_1)$, where $R_1 = R/aR$ and $I_1 = IR_1$. First, if $\operatorname{depth}(R[It]/(a)) = \operatorname{depth}(R_1[I_1t])$, then

$$\begin{aligned}
\operatorname{depth}(R[It]) &= \operatorname{depth}\left(\frac{R[It]}{(a)}\right) + 1 \\
&= \operatorname{depth}(R_1[I_1t]) + 1 \\
&\geq \operatorname{depth}(gr_{I_1}(R_1)) + 1 + 1 \\
&= \operatorname{depth}\left(\frac{gr_I(R)}{(a^*)}\right) + 2 \\
&= \operatorname{depth}(gr_I(R)) - 1 + 2 \\
&= \operatorname{depth}(gr_I(R)) + 1.
\end{aligned}$$

Second, if $\text{depth}(R[It]/(a)) \geq \text{depth}(gr_I(R))$, then

$$\begin{aligned}\text{depth}(R[It]) &= \text{depth}(R[It]/(a)) + 1 \\ &\geq \text{depth}(gr_I(R)) + 1.\end{aligned}$$

Third, if $\text{depth}(gr_I(R)) = \text{depth}\big(R[It]/(a)\big)$, then the assertion is clear. Thus, this completes the proof of case (1).
Case (2) : $r > 0$. Assume that the inequality holds for $r-1$. Since R is normally Cohen-Macaulay along I, we can choose an element $b \in m$ which is a regular element on R/I^{n+1} for all $n \geq 0$ by Lemma 3.3, and hence b is a non-zero-divisor on R and $bR \cap I^n = bI^n$ for all $n \geq 1$ (*cf* : [6], Lemma 1.35). Applying Lemma 2.1, we get the following isomorphism $R[It]/(b) \cong R_2[I_2t]$, where $R_2 = R/bR$ and $I_2 = IR_2$. Hence $\dim\big(R_2/I_2\big) = \dim\big(R/(I,b)\big) = \dim\big(R/I\big) - 1$, and $gr_{I_2}(R_2) \cong gr_I(R)/(b^*)$ is not Cohen-Macaulay, as b^* is a $gr_I(R)$-regular element and R_2 is normally Cohen-Macaulay along I_2. By the inductive hypothesis, we have

$$\begin{aligned}\text{depth}(R_2[I_2t]) &\geq \text{depth}(gr_{I_2}(R_2)) + 1. \\ \text{depth}(R[It]) - 1 &\geq \text{depth}(gr_I(R)) - 1 + 1.\end{aligned}$$

This completes the proof of case (2). ∎

COROLLARY 3.4.1. *([8], Theorem2.1) Let (R,m) be a Cohen-Maca ulay local ring of dimension $d \geq 1$ and I an m-primary ideal. Assume that $gr_I(R)$ is not Cohen-Macaulay. Then*

$$depth(R[It]) = depth\big(gr_I(R)\big) + 1.$$

Proof. : Recall that R is normally Cohen-Macaulay along any m-primary ideal. ∎

We next show that the property of Cohen-Macaulayness of $R[It]$ and $gr_I(R)$ are equivalent for equimultiple ideals by imposing the conditions of a RLR (Regular Local Ring) on R. In other words, using a consequence of the Briançon - Skoda Theorem we can drop the condition $r(I) \leq s-1$ in Theorem 1.2. Recall that an element $a \in R$ is integral over an ideal I if it satisfies an equation of the form

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0, \qquad r_i \in I^i.$$

The set of all elements which are integral over an ideal I form an ideal, denoted by $\overline{I}$ and called the integral closure of I.

REMARKS. : (1) Let R be a Noetherian ring. Then an ideal $J \subseteq I$ is a reduction of I if and only if $I \subseteq \overline{J}$.
(2) The Briançon-Skoda Theorem (see [2], [10], or [7]) states that if (R, m) is a regular local ring and I is an ideal generated by n elements, then $\overline{I^n} \subseteq I$.

LEMMA 3.5. *Let (R, m) be a regular local ring with an infinite residue field and I an equimultiple ideal with $ht(I) = s > 0$. Assume that $gr_I(R)$ is a Cohen-Macaulay ring. Then there exist elements $a_1, \cdots, a_s$ in I such that $I^s = (a_1, \cdots, a_s)I^{s-1}$.*

Proof. : Let $(a_1, \cdots, a_s)$ be a minimal reduction of I. Let $b_1, \cdots, b_r$ be a system of parameters mod$I$, where $r = \dim(R/I) = \dim(R) - ht(I)$. Then $\{ b_1^*, \cdots, b_r^*, a_1^*, \cdots, a_s^* \}$ is a homogeneous system of parameters for $gr_I(R)$, where deg$b_i^* = 0$ for $i = 1, \cdots, r$, and deg$a_j^* = 1$ for $j = 1, \cdots, s$ (cf : [5],Corollary 2.7). Hence it is a $gr_I(R)$-regular sequence since $gr_I(R)$ is Cohen-Macaulay. We have $(a_1, \cdots, a_s) \bigcap I^n = (a_1, \cdots, a_s)I^{n-1}$, $\forall n \geq 1$ (cf : [14],Corollary 2.7). $(a_1, \cdots, a_s)^s$ is a reduction of I^s since $(a_1, \cdots, a_s)$ is a reduction of I. Then

$$(a_1, \cdots, a_s)^s \subseteq I^s \subseteq \overline{(a_1, \cdots, a_s)^s} \subseteq (a_1, \cdots, a_s).$$

Hence we have

$$(a_1, \cdots, a_s)I^{s-1} = (a_1, \cdots, a_s) \bigcap I^s = I^s. \qquad \blacksquare$$

THEOREM 3.6. *Let (R, m) be a regular local ring an infinite residue field and I an equimultiple ideal with $ht(I) = s > 0$. Then the following conditions are equivalent.*
(1) $R[It]$ is a Cohen-Macaulay ring.
(2) $gr_I(R)$ is a Cohen-Macaulay ring.

Proof. : (1) $\Longrightarrow$ (2) : This follows from Proposition 1.1 in [9].
(2) $\Longrightarrow$ (1) : By Lemma 3.5, there exist elements $a_1, \cdots, a_s$ in I such that $I^s = (a_1, \cdots, a_s)I^{s-1}$. This implies $r(I) \leq s - 1$, which proves the assertion from Theorem 1.2. ∎

COROLLARY 3.6.1. *(Huneke, [8], Proposition 2.6) Let (R, m) be a regular local ring* $\dim(R) = d > 0$ *with an infinite residue field and I an m-primary ideal of R. Then $R[It]$ is Cohen-Macaulay if and only if $gr_I(R)$ is Cohen-Macaulay.*

COROLLARY 3.6.2. *Let (R, m) be a regular local ring and I an ideal of R with $ht(I) > 0$. Assume that R is normally Cohen-Macaulay along I. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

Proof. : Case (1) : If $gr_I(R)$ is not Cohen-Macaulay, then we have the equality by Theorem 3.4.
Case (2) : If $gr_I(R)$ is Cohen-Macaulay, then we see that I is equimultiple since R is normally Cohen-Macaulay along I. Hence we have the equality by Theorem 3.6. ∎

COROLLARY 3.6.3. *Let (R, m) be a regular local ring of dimension $d > 0$ and I an m-primary ideal. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

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