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UNSOLVABLE PDO'S OF DEGENERATE TYPE

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1. Introduction

Let A be a linear partial differential operator with C^∞ coefficients in an open set U in R^n. Hörmander has then proved that a necessary condition for local solvability at x_0 is the following:

There exist constants C, k and N such that

$$\left|\int fv\,dx\right| \leq C \sum_{|\alpha\leq k} \sup |D^\alpha f| \sum_{|\beta|\leq N} \sup |D^\beta A^* v|, \tag{1.1}$$

where $f, v \in C_0^\infty(U), U$ is an open set containing the origin.

Here A^* denotes the adjoint of A. Kannai[2] proved that in $R^2_{t,x}, D_t + itD_x^2$ is hypoelliptic but not locally solvable on the line $x = 0$. We consider a real valued C^∞ function $a(t)$ such that $a(t) = t + o(t)$ as $t \to 0$. The purpose of our paper is to show that a partial differential operator

$$\mathcal{A} \equiv D_t + ia(t)D_x^2 \tag{1.2}$$

is still unsolvable on the line $x = 0$, even though we replace t with t + higher order terms. In view of (1.1) it suffices to show that for any open set U containing the origin there exist functions f_λ, v_λ, depending on a real parameter λ and belonging to $C_0^\infty(U)$ such that

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$$\lim_{\lambda\to\infty}\left|\int\int f_\lambda v_\lambda \,dtdx\right| = \infty, \tag{1.3}$$

$$\limsup_{\lambda\to\infty} \sum_{k_1+k_2\le k} \sup\left|D_t^{k_1} D_x^{k_2} f_\lambda\right| < \infty \quad \text{for every } k, \tag{1.4}$$

$$\limsup_{\lambda\to\infty} \sum_{l_1+l_2\le l} \sup\left|D_t^{l_1} D_x^{l_2} \mathcal{A}^* v_\lambda\right| < \infty \quad \text{for every } N, \tag{1.5}$$

2. Main results

Now we state the following:

THEOREM. *The partial differential operator $\mathcal{A}$ is not locally solvable on the line $x = 0$.*

Proof. Let $b(t) = \int_0^t a(y)\,dy$. As in [2] we choose a density function $u_\lambda(t, x)$ of the following form

$$u_\lambda(t,x) = \frac{1}{\sqrt{2b\lambda+1}} \exp\left[\frac{-2b\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b\lambda+1)}\right]. \tag{2.1}$$

as a solution of $\mathcal{A}^*u = 0$. It is enough to show that our operator is not locally solvable at the origin. Let U be an open set containing the origin and $\delta > 0$ a fixed number such that $\{(t,x)|\sqrt{t^2+x^2} < \delta\} \subset U$. We may assume that $2\delta < 1$. From conditions for $a(t), b(t)$ it follows that there exists a constant $c > 0$ such that

$$b(t) = ct^2 + o(t^2), \qquad t \to 0.$$

Thus it is obvious that there exists a constant $c' > 0$ such that $b(t) \ge c't^2$ for t near 0. Without loss of generality we may assume that $c' = 1$. Then we obtain for λ large enough

$$\frac{2b\lambda^2 + x^2\lambda}{2(2b\lambda+1)} \ge \frac{\delta^2\lambda}{2} \tag{2.2}$$

if $\delta \leq |t| \leq 2\delta$ or $\delta \leq |x| \leq 2\delta$. It is clear that for each $k = (k_1, k_2)$

$$
\begin{aligned}
&D_t^{k_1} D_x^{k_2} u_\lambda(t,x) \\
=&G_k(t,x,\lambda,\sqrt{2b\lambda+1})\exp\left[\frac{-2b\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b\lambda+1)}\right]
\end{aligned}
\tag{2.3}
$$

where G_k is a regular function of its arguments because $2b\lambda + 1 > 0$. We consider a function in $C_0^\infty(R^2)$ such that

$$
\phi(t,x) = \begin{cases} 1 & \text{if} \quad \sqrt{t^2+x^2} \leq 1 \\ 0 & \text{if} \quad \sqrt{t^2+x^2} \geq 2. \end{cases}
$$

It follows that there exist constants C_1, C_2 depending on $l = (l_1, l_2)$ such that

$$
\left| D_t^{l_1} D_x^{l_2} \mathcal{A}^* \left[\phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x)\right]\right| \leq C_1 \delta^{-(l_1+l_2+2)} \lambda^{C_2} exp\left[-\frac{\delta^2\lambda}{2}\right] \tag{2.4}
$$

We take a function $F(t,x) \in C_0^\infty(U)$ such that

$$
\int\int F(t,x)\, dtdx = a \neq 0.
$$

It follows that

$$
\lambda^2\, b(\frac{t}{\lambda^2}) \to 0 \qquad \text{as} \qquad \lambda \to \infty.
$$

Thus we obtain

$$
\begin{aligned}
&\lambda^4 \lim_{\lambda\to\infty} \int\int F(\lambda^2 t, \lambda^2 x))\phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x)\, dtdx \\
=&\int\int F(t,x)\phi(0,0)\, dtdx = a.
\end{aligned}
$$

For each fixed k, N we take

$$
\begin{aligned}
f_\lambda(t,x) &= \lambda^{-2k-1} F(\lambda t, \lambda x) \\
v_\lambda(t,x) &= \lambda^{2k+6} \phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x).
\end{aligned}
\tag{2.5}
$$

Moreover it is obvious that

$$
\sum_{k_1+k_2\leq k} \sup |D_t^{k_1} D_x^{k_2} f_\lambda(t,x)| \leq \lambda^{-1}. \tag{2.6}
$$

From (2.2) $\sim$ (2.6) our theorem follows.

COROLLARY. *For each positive integer m the partial differential operator*

$$\mathcal{B} \equiv D_t + i(t^{2m-1} + o(t^{2m-1}))D_x^2$$

is not locally solvable at the origin.

Proof. We sketch briefly the proof. As seen in the proof of Theorem we choose a density function $u_\lambda(t,x)$ of the following form

$$u_\lambda(t,x) = \frac{1}{\sqrt{2b\lambda+1}} \exp\left[\frac{-2b\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b\lambda+1)}\right].$$

as a solution of $\mathcal{B}^* u = 0$, where $b(t) = \int_0^t a(y)\,dy$. We may assume that $b(t) \geq t^{2m}$ for t near 0.

$$\frac{2b\lambda^2 + x^2\lambda}{2(2b\lambda+1)} \geq \frac{\delta^{2m}\lambda}{2}$$

if $\delta \leq |t| \leq 2\delta$ or $\delta \leq |x| \leq 2\delta$. It follows then that there exist constants C_1, C_2 such that

$$\left| D_t^{l_1} D_x^{l_2} \mathcal{B}^* \left[\phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x)\right]\right| \leq C_1 \delta^{-(l_1+l_2+2)} \lambda^{C_2} exp\left[-\frac{\delta^{2m}\lambda}{2}\right]$$

We choose a function $F(t,x) \in C_0^\infty(U)$ such that

$$\int\int F(t,x) exp[-b(t) + ix]\, dtdx = a \neq 0.$$

Then we obtain

$$\lim_{\lambda\to\infty} \lambda^{1+\frac{1}{m}} \int\int F(\lambda^{\frac{1}{m}} t, \lambda x))\phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x)\, dtdx = a.$$

For each fixed k, N we take

$$f_\lambda(t,x) = \lambda^{-k-1} F(\lambda^{\frac{1}{m}} t, \lambda x)$$

$$v_\lambda(t,x) = \lambda^{k+3-\frac{1}{m}} \phi(\frac{t}{\delta}, \frac{x}{\delta}) u_\lambda(t,x).$$

EXAMPLE.

$$D_t + i(t + c\sum_{j=2}^{\infty} t^j)D_x^2$$

is not locally solvable at the origin.

REMARK. For an operator

$$D_t + ia(t)b(t)D_x^2$$

we take a density function

$$u_\lambda(t,x) = \frac{1}{\sqrt{b(t)^2\lambda + 1}} \exp\left[\frac{-b(t)^2\lambda^2 - x^2\lambda + 2ix\lambda}{2(2b(t)\lambda + 1)}\right].$$

We note that the solution of the partial differential equation

$$\mathcal{A}^* u(t,x) = f(t,x)$$
$$u|_{t=0} = 0,$$

might be represented by its partial Fourier transform:

$$\hat{u}(t,\xi) = i\int_0^t \exp\left[-(b(t) - b(s))\xi^2\right] \hat{f}(s,\xi)\, ds.$$

This formula makes sense at least for all $f \in C_0^\infty(R^2)$.

References

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