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DIRECT SUM, SEPARATING SET AND SYSTEMS OF SIMULTANEOUS EQUATIONS IN THE PREDUAL OF AN OPERATOR ALGEBRA

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I. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology coincides with the weak* topology on $\mathcal{L}(\mathcal{H})$(see [3]). Bercovici–Foiaş–Pearcy [3] studied the problem of solving systems of simultaneous equations in the predual of a dual algebra. The theory of dual algebras has been applied to the topics of invariant subspaces, dilation theory and reflexivity (see [1],[2],[3],[5],[6]), and is deeply related with properties $(\mathbf{A}_{m,n})$. Jung–Lee–Lee [7] introduced n–separating sets for subalgebras and proved the relationship between n–separating sets and properties $(\mathbf{A}_{m,n})$. In this paper we will study the relationship between direct sum and properties $(\mathbf{A}_{m,n})$. In particular, using some results of [7] we obtain relationship between n–separating sets and direct sum of von Neumann algebras.

The notation and terminology employed herein agree with those in [3]. The class $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm. The weak* subspace $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the pre–annihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing

$$< T, [L]_{\mathcal{A}} >= tr(TL), \quad T \in \mathcal{A}, \quad [L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}.$$

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We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathcal{H}$, we define a rank one operator $x \otimes y$ by $(x \otimes y)u = (u, y)x$ for all u in $\mathcal{H}$. Throughout this paper, let $\mathbf{N}$ be the set of natural numbers.

DEFINITION 1. Suppose that m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \quad 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For the brevity of notation, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$.

Suppose that $n \in \mathbf{N}$. Let $\mathcal{H}_i$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$ be a dual algebra, $1 \leq i \leq n$. Then we denote the direct sum of dual algebras $\mathcal{A}_i, 1 \leq i \leq n$ by

$$\oplus_{i=1}^n \mathcal{A}_i = \{\oplus_{i=1}^n T_i \in \mathcal{L}(\oplus_{i=1}^n \mathcal{H}_i) | T_i \in \mathcal{A}_i, 1 \leq i \leq n\}.$$

LEMMA 2. *Suppose that $n \in \mathbf{N}$. Let $\mathcal{H}_i$ be a separable, infinite dimensional, complex Hilbert space. Suppose that $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$ is a dual algebra, $1 \leq i \leq n$, with its predual $\mathcal{Q}_{\mathcal{A}_i}$. Then $\oplus_{i=1}^n \mathcal{A}_i \subset \mathcal{L}(\oplus_{i=1}^n \mathcal{H}_i)$ is a dual algebra with its predual $\oplus_{i=1}^n \mathcal{Q}_{\mathcal{A}_i}$ under duality*

$$< \oplus_{i=1}^n T_i, \oplus_{i=1}^n [L_i]_{\mathcal{A}_i} >= \sum_{i=1}^n < T_i, [L_i] >$$

and the norm on $\oplus_{i=1}^n \mathcal{Q}_{\mathcal{A}_i}$ is the norm that accrues to it as a linear manifold in $(\oplus_{i=1}^n \mathcal{A}_i)^*$. In particular, $[(\oplus_{i=1}^n x_i) \otimes (\oplus_{i=1}^n y_i)]$ can be identified with $\oplus_{i=1}^n [x_i \otimes y_i]_{\mathcal{A}_i}$.

Proof. It is easy to show that $\oplus_{i=1}^n \mathcal{A}_i$ is a dual algebra of $\mathcal{L}(\oplus_{i=1}^n \mathcal{H}_i)$. Now, consider the direct sum

$$\oplus_{i=1}^n \mathcal{Q}_{\mathcal{A}_i} = \{\oplus_{i=1}^n [L_i]_{\mathcal{A}_i} | [L_i]_{\mathcal{A}_i} \in \mathcal{Q}_{\mathcal{A}_i}\}$$

of Banach spaces $\mathcal{Q}_{\mathcal{A}_i}, 1 \leq i \leq n$, with the usual direct sum norm.

For $\oplus_{i=1}^{n} T_i \in \oplus_{i=1}^{n} \mathcal{A}_i$ and $\oplus_{i=1}^{n} [L_i]_{\mathcal{A}_i} \in \oplus_{i=1}^{n} \mathcal{Q}_{\mathcal{A}_i}$, we define

$$< \oplus_{i=1}^{n} T_i, \oplus_{i=1}^{n} [L_i]_{\mathcal{A}_i} >= \sum_{i=1}^{n} < T_i, [L_i]_{\mathcal{A}_i} > .$$

Then it is easy to show that $< \cdot, \oplus_{i=1}^{n} [L_i]_{\mathcal{A}_i} >$ defines a linear functional on $\oplus_{i=1}^{n} \mathcal{A}_i$, which we may define by $\oplus_{i=1}^{n} [L_i]$. We define $|| \oplus_{i=1}^{n} [L_i]||$ to be the norm of this linear functional. Since $\oplus_{i=1}^{n} [L_i]$ is ultraweakly continuous on $\oplus_{i=1}^{n} \mathcal{A}_i$ by [4, Problem 15.J], $\oplus_{i=1}^{n} [L_i]$ corresponds to an element of the predual $\mathcal{Q}_{\oplus_{i=1}^{n} \mathcal{A}_i}$.

On the other hand, if $[L] \in \mathcal{Q}_{\oplus_{i=1}^{n} \mathcal{A}_i}$, we write

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix} \in \mathcal{L}(\oplus_{i=1}^{n} \mathcal{H}_i).$$

Furthermore, since $\oplus_{i=1}^{n} \mathcal{A}_i \in \mathcal{L}(\oplus_{i=1}^{n} \mathcal{H}_i)$, we may define a linear functional on $\oplus_{i=1}^{n} \mathcal{A}_i$ such that

$$< \oplus_{i=1}^{n} A_i, [L] >= tr(A_{i_0} L_{i_0 i_0}).$$

Letting i_0 range over the set $\{1, 2, \cdots, n\}$, we obtain an element $\oplus_{i=1}^{n} [L_i]$ corresponding to $[L]$ and

$$< \oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} [L_i] >= \sum_{i=1}^{n} < A_i, [L_i] > .$$

Finally, for any $\oplus_{i=1}^{n} T_i \in \oplus_{i=1}^{n} \mathcal{A}_i$, we have

$$< \oplus_{i=1}^{n} T_i, [(\oplus_{i=1}^{n} x_i) \otimes (\oplus_{i=1}^{n} y_i)] >=< \oplus_{i=1}^{n} T_i, \oplus_{i=1}^{n} [x_i \otimes y_i] > .$$

The proof is complete.

The following lemma comes from Proposition 2.04 of [3].

LEMMA 3. *If $\mathcal{A}$ is a dual algebra with properties $(\mathbf{A}_{m,n})$ for some $1 \le m, n \le \aleph_0$ and $\mathcal{B}$ is any subalgebra of $\mathcal{A}$, then $\mathcal{B}$ has the same property.*

The following theorem should be compared with Proposition 1.3 of [1] and Proposition 2.055 of [3].

THEOREM 4. *Suppose that m, n are cardinal numbers such that $1 \le m, n \le \aleph_0$ and $p \in \mathbf{N}$. Let $\mathcal{A}_k$ be a dual algebra, $1 \le k \le p$. Then $\mathcal{A}_k$ has property $(\mathbf{A}_{m,n})$ for any $1 \le k \le p$ if and only if $\oplus_{k=1}^{p} \mathcal{A}_k$ has property $(\mathbf{A}_{m,n})$.*

Proof. We shall prove this theorem when $1 \le m, n < \aleph_0$. Let $\oplus_{k=1}^{p}[L_{ij}^{(k)}] \in \oplus_{k=1}^{p} \mathcal{Q}_{\mathcal{A}_k}$. Then there exist sequences $\{x_i^{(k)}\}_{i=1}^{m}$ and $\{y_j^{(k)}\}_{j=1}^{n}$ in $\mathcal{H}_k$ such that $[L_{ij}^{(k)}]_{\mathcal{A}_k} = [x_i^{(k)} \otimes y_j^{(k)}]_{\mathcal{A}_k}$ for each $1 \le k \le p$. Now let us set

$$\tilde{x}_i = x_i^{(1)} \oplus x_i^{(2)} \oplus \cdots \oplus x_i^{(p)},$$

$$\tilde{y}_j = y_j^{(1)} \oplus y_j^{(2)} \oplus \cdots \oplus y_j^{(p)}.$$

Then it is obvious that $\tilde{x}_i, \tilde{y}_j \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p, 1 \le i \le m, 1 \le j \le n$. Furthermore, according to Lemma 2 we have

$$\oplus_{k=1}^{p}[L_{ij}^{(k)}] = \oplus_{k=1}^{p}[x_i^{(k)} \otimes y_j^{(k)}] = [\tilde{x}_i \otimes \tilde{y}_j].$$

Finally, we can solve the required simultaneous systems for the statement when of $n = \aleph_0$ or $m = \aleph_0$ by a similar method with the above.

The converse is obvious by Lemma 3, and the proof is complete.

Now, we consider the countable direct sum of dual algebras, i.e.,

$$\oplus_{i=1}^{\infty} \mathcal{A}_i = \{\oplus_{i=1}^{\infty} T_i \in \mathcal{L}(\widetilde{\mathcal{H}}) | T_i \in \mathcal{A}_i, \sup \|T_i\| < \infty\}$$

where $\widetilde{\mathcal{H}} = \{\oplus_{i=1}^{\infty} x_i \in \oplus_{i=1}^{\infty} \mathcal{H}_i | \sum \|x_i\|^2 < \infty\}$. It is obvious that if $\oplus_{i=1}^{\infty} \mathcal{A}_i$ has property $(\mathbf{A}_{m,n})$ for $1 \le m, n \le \aleph_0$, then $\mathcal{A}_i$ has property $(\mathbf{A}_{m,n})$ for all i.

DEFINITION 5. [7]. Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$ and let $\{x_i\}_{i=1}^n$ be a linearly independent subset of $\mathcal{H}, n \in \mathbb{N}$. Then $\{x_i\}_{i=1}^n$ is said to be an *n–separating set* for $\mathcal{A}$ if $\sum_{i=1}^n T_i x_i = 0$ for $T_i \in \mathcal{A}$ implies $T_i = 0, 1 \leq i \leq n$. And we say that $\mathcal{A}$ has an *n–separating set* $\{x_i\}_{i=1}^n$.

Note that an algebra with an n–separating set has an m–separating set for $m \leq n$.

REMARK. Let $\mathcal{A}_i$ be a dual algebra of $\mathcal{L}(\mathcal{H}_i), i = 1, 2, \cdots$. We claim that $\oplus_{i=1}^\infty \mathcal{A}_i$ can be considered as a subspace of $\mathcal{L}(\widetilde{\mathcal{H}})$ under the weak* topology on $\mathcal{L}(\widetilde{\mathcal{H}})$. To do so, let $\oplus_{i=1}^\infty T_i^{(\alpha)}$ be a net converging to an operator $R \in \mathcal{L}(\widetilde{\mathcal{H}})$ under the weak* topology on $\mathcal{L}(\widetilde{\mathcal{H}})$. Then

$$\sum_{k=1}^{\infty}(\oplus_{i=1}^\infty T_i^{(\alpha)} \tilde{x}^{(k)}, \tilde{y}^{(k)}) \to \sum_{k=1}^{\infty}(\oplus_{i=1}^\infty R\tilde{x}^{(k)}, \tilde{y}^{(k)}) \qquad (*)$$

for any square summable sequences $\{\tilde{x}^{(k)}\}_{k=1}^\infty$ and $\{\tilde{y}^{(k)}\}_{k=1}^\infty$ in $\widetilde{\mathcal{H}}$. Let us write

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots \\ R_{21} & R_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

relative to $\widetilde{\mathcal{H}}$. Let us denote

$$\tilde{x}^{(k)} = x_1^{(k)} \oplus x_2^{(k)} \oplus \cdots ,$$

$$\tilde{y}^{(k)} = y_1^{(k)} \oplus y_2^{(k)} \oplus \cdots .$$

Now we take square summable sequence $\{x_i^{(k)}\}_{k=1}^\infty$ in $\mathcal{H}_i$ and $\{y_j^{(k)}\}_{k=1}^\infty$ in $\mathcal{H}_j, i, j = 1, 2, \cdots$. Let us set

$$\tilde{x}_i^{(k)} = \overbrace{0 \oplus \cdots \oplus 0}^{(i-1)} \oplus x_i^{(k)} \oplus 0 \oplus \cdots ,$$

$$\tilde{y}_j^{(k)} = \overbrace{0 \oplus \cdots \oplus 0}^{(j-1)} \oplus y_j^{(k)} \oplus 0 \oplus \cdots .$$

Substituting $\{\tilde{x}_i^{(k)}\}_{k=1}^{\infty}$ and $\{\tilde{y}_j^{(k)}\}_{k=1}^{\infty}$ in (*), we have

$$\sum_{k=1}^{\infty}((\oplus_{l=1}^{\infty}T_l^{(\alpha)})\tilde{x}_i^{(k)}, \tilde{y}_j^{(k)}) \to \sum_{k=1}^{\infty}(R_{1i}x_i^{(k)} \oplus R_{2i}x_i^{(k)} \oplus \cdots, \tilde{y}_j^{(k)}),$$

for any $i, j = 1, 2, \cdots$. It is easy to show that

$$R_{ji} = 0, j \neq i.$$

Hence $R = \oplus_{i=1}^{\infty}R_{ii}$. Furthermore, we have that

$$\sum_{k=1}^{\infty}(T_i^{(\alpha)}x_i^{(k)}, y_i^{(k)}) \to \sum_{k=1}^{\infty}(R_{ii}x_i^{(k)}, y_i^{(k)})$$

for any $i = 1, 2, \cdots$. Since $\mathcal{A}_i$ is weak* closed, $R_{ii} \in \mathcal{A}_i$. So $R \in \oplus_{i=1}^{\infty}\mathcal{A}_i$. Therefore $\oplus_{i=1}^{\infty}\mathcal{A}_i$ is a dual algebra in $\mathcal{L}(\widetilde{\mathcal{H}})$.

THEOREM 6. *Suppose that $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$ is a dual algebra with a k_i–separating set in $\mathcal{H}_i$ for $k_i \in \mathrm{N}, i = 1, 2, \cdots$. Let $m = min\{k_i\}$. Then the dual algebra $\oplus_{i=1}^{\infty}\mathcal{A}_i$ has an m-separating set in $\widetilde{\mathcal{H}}$.*

Proof. For each i, let $\{x_k^{(i)}\}_{k=1}^{k_i}$ be a k_i–separating set for $\mathcal{A}_i$ in $\mathcal{H}_i$. Consider a positive real number

$$M_{k,i} = \frac{1}{2^i(1 + \|x_k^{(i)}\|)}$$

for $1 \leq k \leq m, i = 1, 2, \cdots$. Let $\tilde{x}_j = \oplus_{i=1}^{\infty}M_{j,i}x_j^{(i)}, 1 \leq i \leq m$. Then $\tilde{x}_j \in \widetilde{\mathcal{H}}$. An easy calculation shows that $\{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m\}$ is linearly independent.

Suppose that $\sum_{k=1}^{m}(\oplus_{i=1}^{\infty}T_i^{(k)})\tilde{x}_k = 0$ for any $\oplus_{i=1}^{\infty}T_i^{(k)} \in \oplus_{i=1}^{\infty}\mathcal{A}_i, 1 \leq k \leq m$. Since

$$\begin{aligned}\sum_{k=1}^{m}(\oplus_{i=1}^{\infty}T_i^{(k)})\tilde{x}_k &= \sum_{k=1}^{m}(\oplus_{i=1}^{\infty}T_i^{(k)})(\oplus_{i=1}^{\infty}M_{k,j}x_k^{(j)}) \\ &= \sum_{k=1}^{m}\oplus_{i=1}^{\infty}T_i^{(k)}(M_{k,i}x_k^{(i)}) \\ &= \oplus_{i=1}^{\infty}\sum_{k=1}^{m}T_i^{(k)}(M_{k,i}x_k^{(i)}),\end{aligned}$$

we have $\sum_{k=1}^{m} M_{k,i}T_I^{(k)}x_k^{(i)} = 0, i = 1, 2, \cdots$. Since $\{x_k^{(i)}\}_{k=1}^{k_i}$ is an m-separating set for $\mathcal{A}_i, M_{k,i}T_i^{(k)} = 0, 1 \le k \le m$. Thus $T_i^{(k)} = 0, 1 \le k \le m, i = 1, 2, \cdots$ and hence $\oplus_{i=1}^{\infty} T_i^{(k)} = 0, 1 \le k \le m$. Hence $\{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m\}$ is an m-separating set for $\oplus_{i=1}^{\infty}\mathcal{A}_i$.

REMARK. The condition of minimality of k_i in appearing in Theorem 6 is essential (for example, consider algebras generated by T_1, T_2, and $T_1 \oplus T_2$, where $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$).

The following is an immediate corollary of Theorem 6.

COROLLARY 7. *The dual algebra $\mathcal{A}_i$ has an n-separating set in $\mathcal{H}_i$, if and only if the dual algebra $\oplus_{i=1}^{\infty}\mathcal{A}_i$ has an n-separating set in $\widetilde{\mathcal{H}}$.*

The following lemma plays a central role for one of the main results in this paper.

LEMMA 8. *[7],[8]. Suppose that n is a cardinal number such that $1 \le n \le \aleph_0$. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and let $n \in \mathbf{N}$. Then the following are equivalent:*

(1) *$\mathcal{A}$ has property $(\mathbf{A}_n)$.*
(2) *$\mathcal{A}$ has an n-separating set.*
(3) *$\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$.*

Finally, we consider some necessary and sufficient conditions for the direct sum of von Neumann algebras with property $(\mathbf{A}_{m,n})$.

THEOREM 9. *Suppose that $\mathcal{A}_i$ is a von Neumann algebra for $i = 1, 2, \cdots$. Then the following are equivalent:*

(1) *$\mathcal{A}_i$ has an n-separating set for all $i = 1, 2, \cdots$.*
(2) *$\oplus\mathcal{A}_i$ has an n-separating set.*
(3) *$\oplus\mathcal{A}_i$ has property $(\mathbf{A}_{n,\aleph_0})$.*
(4) *$\mathcal{A}_i$ has property $(\mathbf{A}_{n,\aleph_0})$ for all $i = 1, 2, \cdots$.*
(5) *$\mathcal{A}_i$ has property $(\mathbf{A}_n)$ for all $i = 1, 2, \cdots$.*
(6) *$\oplus\mathcal{A}_i$ has property $(\mathbf{A}_n)$.*

Proof. The proof is clearly by Corollary 7 and Lemma 8.

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