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A REMARK ON MULTIPLICATION MODULES

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Modules which satisfy the converse of Schur's lemma have been studied by many authors. In [6], R. Ware proved that a projective module P over a semiprime ring R is irreducible if and only if $End_R(P)$ is a division ring. Also, Y. Hirano and J.K. Park proved that a torsionless module M over a semiprime ring R is irreducible if and only if $End_R(M)$ is a division ring. In case R is a commutative ring, we obtain the following: An R-module M is irreducible if and only if $End_R(M)$ is a divison ring and M is a multiplication R-module.

Throughout this paper, R is a commutative ring with identity and all modules are unital left R-modules.

Let R be a commutative ring with identity and let M be an R-module. Then M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$.

Cyclic R-modules are multiplication modules. In particular, irreducible R-modules are multiplication modules.

An endomorphism f of an R-module M is called trivial if there exists $a \in R$ such that $f(m) = am$, for all $m \in M$. We set

$$Tri(M) = \{f \in End_R(M) | f \text{ is trivial } \}.$$

Clearly, $Tri(M) \cong R/ann_R(M)$ and $Tri(M)$ is a subring of $End_R(M)$.

THEOREM 1. *Let M be a finitely generated faithful multiplication R-module. If $End_R(M)$ is a divison ring, then R is a field and M is an 1-dimensional vector space over R.*

Proof. Let M be a finitely generated faithful multiplication R-module. Then $R \cong Tri(M)$, and R can be embedded in $End_R(M)$. Since M

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is finitely generated multiplication R-module, $End_R(M) = Tri(M)$ [5, Theorem 3]. Hence $R \cong End_R(M)$. Thus R is a field. In view of Corollary in [3], M is an 1-dimensional vector space over R.

LEMMA 2. *Let M be a faithful R-module. Then M is irreducible if and only if $End_R(M)$ is a division ring and M is a multiplication R-module.*

Proof. It remains to show the "if" part. Assume that M is a faithful multiplication R-module and let $End_R(M)$ be a division ring. Since M be a faithful R-module, R can be embedded in $End_R(M)$. Since $End_R(M)$ is a division ring, R is an integral domain. Thus R is a semiprime ring. Now M is a faithful multiplication R-module, M is torsionless [4, Theorem 1.3]. By [2, Corollary 3], M is irreducible.

THEOREM 3. *Let M be an R-module. Then M is irreducible if and only if $End_R(M)$ is a division ring and M is a multiplication R-module.*

Proof. Let $I = ann_R(M)$. Then M is a faithful R/I-module and M is a multiplication R/I-module. Since $End_{R/I}(M) = End_R(M)$ and $End_R(M)$ is a division ring, $End_{R/I}(M)$ is a division ring. By Lemma 2, M is a irreducible R/I-module. Thus M is a irreducible R-module.

COROLLARY. *Let M be a projective R-module. Then M is irreducible if and only if $End_R(M)$ is a divison ring.*

Proof. If M is a projective R-module and $End_R(M)$ is a divison ring, then M is a cyclic module [6, Proposition 4.3]. Thus M is a multiplication R-module. By Theorem 3, M is irreducible.

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