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CO–CLOSED SURFACES OF 1–TYPE GAUSS MAP

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1. Introduction

Submanifolds of finite type were introduced by B.-Y. Chen ([2]) about ten years ago. Many works have been done in characterizing or classifying submanifolds in Euclidean space with this notion. On the other hand, the Gauss map is one of the most useful tools to study submanifolds of Euclidean space. In a work of finite type theory, B.-Y. Chen and P. Piccini ([3]) studied compact submanifolds with finite type Gauss map. In general, non-compact manifolds are usually difficult to be handled. In this article we are going to deal with a part of the following most general question: What kind of surfaces in 3-dimensional Euclidean space have 1-type Gauss map?

A submanifold M of Euclidean m-space E^m is said to be of finite type ([2]) if each component of its position vector X can be represented as a finite sum of eigenfunctions of Laplace-Beltrami operator Δ of M with respect to the induced metric from that of E^m, that is,

$$X = X_0 + X_1 + \cdots + X_k$$

where X_0 is a constant map, $X_1, \cdots, X_k$ are non-constant maps satisfying $\Delta X_i = l_i X_i, l_i$ being constants, $i = 1, 2, \cdots, k$. In particular, if $l_1, \cdots, l_k$ are mutually different, we say that M is of k-type. Similarly, a smooth map ϕ of a Riemannian manifold M into E^m is said to be of finite type if ϕ is a finite sum of E^m-valued eigenfunctions of Δ(ϕ is not necessarily isometric).

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2. Co-closed surfaces

Let M be a surface in E^3 and e_1, e_2 principal directions of M. We denote by ω^1, ω^2 their dual 1-forms. We then have the connection forms ω_i^j of M defined by $\nabla_{e_i} e_j = \sum_k \omega_j^k(e_i)e_k$ where ∇ is the Levi-Civita connection induced from that of E^3. We now give the definition of co-closed surface.

DEFINITION. A surface of Euclidean 3-space is called co-closed if the connection form ω_1^2 is co-closed, that is, trace $(\nabla\omega_1^2) = 0$.

REMARK. (Geometric meaning of co-closed surfaces). Let M be a co-closed surface and δ the co-differential operator of M. Then $\delta\omega_1^2 = 0$, where ω_1^2 is the connection form in the definition above. Let $\omega_1^2 = f\omega^1 + g\omega^2$, where f and g are smooth functions. Then $\delta\omega_1^2 = - * d * \omega_1^2 = 0$ if and only if $e_1 f + e_2 g = 0$, where $*$ denotes the Hodge-star operator. Since $*\omega_1^2 = f\omega^2 - g\omega^1, d * \omega_1^2 = 0$ implies that $g\omega^1 - f\omega^2 = du$ locally for some function u, that is, $\omega_1^2 = -(e_2 u)\omega^1 + (e_1 u)\omega^2$. The structure equation provides that

$$\begin{aligned} -K\omega^1 \wedge \omega^2 = d\omega_1^2 &= \{e_1 e_1 u + e_2 e_2 u + (e_1 u)^2 + (e_2 u)^2\}\omega^1 \wedge \omega^2 \\ &= -(\Delta u)\omega^1 \wedge \omega^2, \end{aligned}$$

which implies

$$K = \Delta u,$$

where K is the Gauss curvature of M, that is, the Gauss curvature can be locally expressed as Laplacian of some function.

EXAMPLE 1. Every flat surface is co-closed.

EXAMPLE 2.. Surfaces of revolution are co-closed. In fact, let M be a surface of revolution in E^3 parametrized (locally) by $x(s,t) = (f(s)\cos t, f(s)\sin t, g(s))$ satisfying $(f'(s))^2 + (g'(s))^2 = 1$ and $f(s) > 0$. Then the metric tensor is given by $\begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix}$ and thus the Christoffel symbols are given by $\Gamma^1_{22} = -ff', \Gamma^2_{12} = \frac{f'}{f}$ and others are zero. We can choose $e_1 = \frac{\partial x}{\partial s}, e_2 = \frac{1}{f}\frac{\partial x}{\partial t}$ which are principal directions. In this case $\omega^2_1 = \frac{f'}{f}\omega^2$. We can easily check that $(\nabla_{e_1}\omega^2_1)(e_1) + (\nabla_{e_2}\omega^2_1)(e_2) = 0$.

3. Co-closed surfaces in E^3 with 1-type Gauss map

Let M be a surface of E^3 with 1-type Gauss map and G the Gauss map of M into $G(2,3)$ which is the Grassmann manifold of the oriented 2-planes in E^3. $G(2,3)$ can be identified with the decomposable 2-vectors of norm 1 in 3-dimensional Euclidean space $\Lambda^2 E^3 \cong E^3$. Let e_1 and e_2 be an oriented orthonormal frame on M. Then, $G : M \longrightarrow G(2,3)$ can be given by $G(p) = (e_1 \wedge e_2)(p), p \in M$. Likewise Theorem 2.1 and Theorem 2.2 in [3] there exist a constant a and a constant vector $\mathbf{c}$ such that

$$(1) \qquad \Delta G - a(G - \mathbf{c}) = 0$$

where Δ is the Laplace operator on M. We now choose e_1 and e_2 as principal directions of M and let x and y be the corresponding principal curvatures of the shape operator A of M. Then (1) implies

$$(2) \quad (e_2x + e_2y)e_1 \wedge e_3 + (e_1x + e_2y)e_3 \wedge e_2 - (x^2 + y^2 - a)e_1 \wedge e_2 = a\mathbf{c},$$

where e_3 is the unit normal vector field to M. Let $\omega^1, \omega^2, \omega^3$ be the dual 1-forms to e_1, e_2 and e_3 and ω^B_A the connection forms associated with $\omega^1, \omega^2, \omega^3$ satisfying $\omega^B_A + \omega^A_B = 0$ and

$$(3) \qquad \tilde{\nabla}_{e_i} e_j = \sum_k \omega^k_j(e_i)e_k + h_{ji}e_3, \nabla_{e_i} e_j = \sum_k \omega^k_j(e_i)e_k,$$

$$\tilde{\nabla}_{e_i} e_3 = \sum_k \omega_3^k(e_i) e_k, \tag{4}$$

$$x = \omega_1^3(e_1) = h_{11}, y = \omega_2^3(e_2) = h_{22}, h_{12} = h_{21} = 0, \tag{5}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections of E^3 and M respectively and h the second fundamental form of M. The indices A, B run over the range $\{1,2,3\}$ and i,j,k over $\{1,2\}$. Let X be a vector field tangent to M. Taking the covariant differentiation of (2.2), we have

$$\begin{aligned}
&\{X(e_2x) + X(e_2y) + (e_1x + e_1y)\omega_1^2(X) - (\|h\|^2 - a)\omega_2^3(X)\}e_1 \wedge e_3 \\
&+ \{(e_2x + e_2y)\omega_1^2(X) - X(e_1x + e_1y) + (\|h\|^2 - a)\omega_1^3(X)\}e_2 \wedge e_3 \\
&+ \{(e_2x + e_2y)\omega_3^2(X) + (e_1x + e_1y)\omega_3^1(X) - X\|h\|^2\}e_1 \wedge e_2 = 0.
\end{aligned}$$

If we take X as e_1, then we obtain

$$e_1e_2(x+y) + e_1(x+y)\omega_1^2(e_1) = 0, \tag{6}$$

$$e_2(x+y)\omega_1^2(e_1) - e_1e_1(x+y) + x(\|h\|^2 - a) = 0, \tag{7}$$

$$xe_1(x+y) + e_1\|h\|^2 = 0. \tag{8}$$

Similarly, taking X as e_2, we get

$$e_2e_2(x+y) + e_1(x+y)\omega_1^2(e_2) - y(\|h\|^2 - a) = 0, \tag{9}$$

$$e_2(x+y)\omega_1^2(e_2) - e_2e_1(x+y) = 0, \tag{10}$$

$$ye_2(x+y) + e_2\|h\|^2 = 0. \tag{11}$$

From (6)-(11), we have the following

PROPOSITION 3.1. *Let M be a surface of 3-dimensional Euclidean space with 1-type Gauss map. If the mean curvature is constant, then M is a part of plane or a part of circular cylinder or a part of sphere.*

Proof. We see that $\|h\|^2$ is constant from (8) and (11). The proposition immediately follows from the proposition in pp. 118, [1]. ||||

From now on, we assume that M is connected co-closed. (8) and (11) imply that

$$3xe_1x + (x+2y)e_1y = 0, \tag{12}$$

$$(2x+y)e_2x + 3ye_2y = 0 \tag{13}$$

Using the structure equation: $d\omega_i^3 = \sum_j \omega_i^j \wedge \omega_j^3$, we obtain

$$e_2x = (y-x)\omega_2^1(e_1), \tag{14}$$

$$e_1y = (x-y)\omega_1^2(e_2), \tag{15}$$

since $\omega_1^3 = x\omega^1, \omega_2^3 = y\omega^2$ and $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$.

Consider a subset $M_0 = \{p \in M | x(p) \neq 0, y(p) \neq 0, x(p) \neq y(p)\}$. Suppose $M_0 \neq \emptyset$. On M_0, (12)-(15) imply

$$e_1x = -\frac{(x+2y)(x-y)}{3x}\omega_1^2(e_2), \tag{16}$$

$$e_2y = \frac{(y+2x)(x-y)}{3y}\omega_2^1(e_1), \tag{17}$$

Substituting (14) and (17) into (6) and making use of (15) and (16), we have after a long computation

$$(\nabla_{e_1}\omega_2^1)(e_1) = \frac{3x^2+2xy+7y^2}{3xy}\omega_2^1(e_1)\omega_1^2(e_2), \tag{18}$$

where $(\nabla_{e_1}\omega_2^1)(e_1) = e_1(\omega_2^1(e_1)) - \omega_2^1(e_2)\omega_1^2(e_1)$. Putting (14) and (17) into (9), we obtain

(19)
$$(\nabla_{e_2}\omega_2^1)(e_1) = -\frac{3y^2}{2(x-y)}(a - x^2 - y^2) + \frac{x-y}{x}(\omega_1^2(e_2))^2 + \frac{2x^2+3xy+7y^2}{3y^2}(\omega_2^1(e_1))^2,$$

where $(\nabla_{e_2}\omega_2^1)(e_1) = e_2(\omega_2^1(e_1)) + (\omega_2^1(e_2))^2$. Similarly, (7) and (14)-(17) produce

(20)
$$(\nabla_{e_1}\omega_1^2)(e_2) = -\frac{3x^2}{2(x-y)^2}(a - x^2 - y^2) - \frac{x-y}{y}(\omega_2^1(e_1))^2 + \frac{7x^2+3xy+2y^2}{3x^2}(\omega_1^2(e_2))^2,$$

where $(\nabla_{e_1}\omega_1^2)(e_2) = e_1(\omega_1^2(e_2)) - (\omega_2^1(e_1))^2$. Also, (10) together with (14)-(17) yields

(21)
$$(\nabla_{e_2}\omega_1^2)(e_2) = \frac{7x^2+2xy+3y^2}{3xy}\omega_2^1(e_1)\omega_1^2(e_2),$$

where $(\nabla_{e_2}\omega_1^2)(e_2) = e_2(\omega_1^2(e_2)) - \omega_1^2(e_1)\omega_2^1(e_2)$.

Since M is co-closed, (18) and (21) implies

$$-\frac{4(x-y)(x+y)}{3xy}\omega_2^1(e_1)\omega_1^2(e_2) = 0.$$

It follows that

(22)
$$(x+y)\omega_2^1(e_1)\omega_1^2(e_1) = 0 \text{ on } M_0.$$

Let $M_1 = \{p \in M_0 | \omega_2^1(e_1)\omega_1^2(e_2) \neq 0\}$. Suppose that M_1 is not empty. Then $x + y = 0$ on M_1. (7) and (9) imply that $\|h\|^2 = x^2 + y^2 = a$. Differentiating this covariantly with respect to e_1 and making use of

(15) and (16), we obtain $-\frac{(x-y)^2}{3}\omega_1^2(e_2) = 0$, which is a contradiction. Thus, M_1 is empty and $\omega_2^1(e_1)\omega_1^2(e_2) = 0$ on M_0. Consider the set $M_2 = \{p \in M_0 | \omega_1^2(e_2) \neq 0\}$. Suppose $M_2 \neq \emptyset$. Then $\omega_2^1(e_1) = 0$ on M_2. Thus, $e_2 x = e_2 y = 0$ on the open set M_2. On M_2, (7) and (8) become

$$e_1 e_1(x+y) = x(x^2+y^2-a), \tag{23}$$

$$x e_1(x+y) + e_1(x^2+y^2) = 0. \tag{24}$$

Since $0 = e_2(\omega_2^1(e_1)) = (\nabla_{e_2}\omega_2^1)(e_1) - (\omega_2^1(e_2))^2$, it follows that

$$(\omega_2^1(e_2))^2 = \frac{3xy}{2(x-y)^2}(x^2+y^2-a) \text{ on } M_2. \tag{25}$$

Differentiating (24) covariantly with respect to e_1 and making use of (9) with $e_2 x = e_2 y = 0$, (21) and (25), we get

$$\frac{y(x+2y)}{6x(x-y)}(a-x^2-y^2)(2x^2-4xy+3axy-3x^3y+2y^2-3xy^3) = 0,$$

or, since $y \neq 0$ and $x^2+y^2-a \neq 0$,

$$(x+2y)\{2(x-y)^2+3xy(a-x^2-y^2)\} = 0 \text{ on } M_2.$$

Let $M_3 = \{p \in M_2 | (x+2y)(p) \neq 0\}$. Suppose $M_3 \neq \emptyset$. Then, $2(x-y)^2+3xy(a-x^2-y^2) = 0$ on M_3, which and (25) imply

$$(\omega_2^1(e_2))^2 = 1 \text{ on } M_3.$$

It follows that $(\nabla_{e_1}\omega_1^2)(e_2) = 0$ on M_3. On the other hand, (19) and (25) imply

$$(\nabla_{e_2}\omega_2^1)(e_1) = 1.$$

Since $-K = d\omega_1^2(e_1, e_2) = (\nabla_{e_1}\omega_1^2)(e_2) - (\nabla_{e_2}\omega_2^1)(e_1)$ where K is the Gauss curvature of M, $K = xy = -1$. Differentiating this covariantly with respect to e_1 and using (15) and (16), we get $3x^2 - xy - 2y^2 = 0$

on M_3. It gives $y = -\frac{3}{2}x$ on M_3. Since $xy = -1, x$ and y are constant on M_3, which contradicts $e_1 x \neq 0, e_1 y \neq 0$ on M_3. Therefore, $M_3 = \emptyset$, that is, $x + 2y = 0$ on M_2. Differentiating this with respect to e_1 and using (15) and (16), we get $x = y$ which is a contradiction.

Hence M_2 must be empty, that is, $\omega_1^2(e_2) = 0$ on M_0. By developing the same argument before, we see that $\omega_1^2(e_1) = 0$ on M_0. Thus, x and y are constant on each component of M_0. By proposition 3.1, a component of M_0 is a part of plane, a circular cylinder or a sphere. However, it is impossible on M_0. Therefore, $M_0 = \emptyset$ and M can be expressed as $N_1 \cup N_2 \cup N_3$ where $N_1 = \{p \in M | x(p) = 0\}$, $N_2 = \{p \in M | y(p) = 0\}$ and $N_3 = \{p \in M | x(p) = y(p) \neq 0\}$. Suppose that $Int(N_1) \neq \emptyset$, where $Int(N_1)$ means the interior of N_1. (12) and (13) imply that y is constant on each component of $Int(N_1)$. On a component C_1 of $Int(N_1)$, the mean curvature and the length of second fundamental form are constant on C_1. Hence, C_1 is contained in a plane or a circular cylinder by the proposition, pp. 118 in [1]. By continuity, C_1 is also closed and so C_1 is M. We have the same situatiation as the previous case if $Int(N_2) \neq \emptyset$. Suppose $Int(N_3) \neq \emptyset$. Let C_3 be a component of N_3. We can easily show that x is constant on C_3. Thus C_3 is contained in a sphere. By continuity, C_3 is closed. Consequently, $C_3 = M$.

Thus, we have

THEOREM 3.2. *Planes, spheres and circular cylinders are the only co-closed connected surfaces in E^3 with 1-type Gauss map.*

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