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ON RELATIVE CHINESE REMAINDER THEOREM

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Previously T. Porter [3] has given a relative Chinese Remainder Theorem under the hypothesis that given ring R has at least one τ-closed maximal ideal (by his notation $\mathrm{Max}_\tau(R) \neq \phi$). In this short paper we drop his overall hypothesis that $\mathrm{Max}_\tau(R) \neq \phi$ and give the proof and some related results with this Theorem.

In this paper R will always denote a commutative ring with identity element and all modules will be unitary left R-modules unless otherwise specified.

Let τ be a given hereditarty torsion theory for left R-module category R-Mod. The class of all τ-torsion left R-modules, dented by $\mathcal{J}$ is closed under homomorphic images, submodules, direct sums and extensions. And the class of all τ-torsionfree left R-modules, denoted by $\mathcal{F}$, is closed under taking submodules, injective hulls, direct products, and isomorphic copies([2], Proposition 1.7 and 1.10).

Notation and terminology concerning (hereditaty) torsion theories on R-Mod will follow [2]. In particular, if τ is a torsion theory on R-Mod, then a left R-submodule N of M is said to be τ-closed (τ-dense, resp.) submodule of M if and only if M/N is τ-torsionfree (τ-torsion, resp.). A module M is called τ-cocritical if $M \in \mathcal{F}$ and $M/N \in \mathcal{J}$ for each nonzero submodule N of M. A left ideal L of R is τ-critical if R/L is τ-cocritical.

Follow Porter [3], we denote $\mathrm{Max}_\tau(M)$ be the set of all maximal τ-closed submodules of M and we say ideals I, J are *τ-comaximal* if $I + J$ is τ-dense in R. Let $I_1, I_2, \cdots, I_n$ be ideals of R, they are *pairwise τ-comaximal* in case $I_i + I_j$ is τ-dense in R whenever $i \neq j$. For example, if each I_i is a maximal τ-closed ideal of R or each I_i is a τ-critical ideal, then these ideals are pairwise τ-comaximal.

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The following Lemma 1 and Theorem 2 can be found in [3], we give the proof of Lemma 1 for the completeness of this paper.

LEMMA 1. *(Porter, [3]) Let M be a left R-module, and I, J be τ-comaximal ideals in R, then $(IM \cap JM)/IJM$ is τ-torsion.*

Proof. If $x \in IM \cap JM$, $(I + J)x \in IJM$. Since $I + J$ is τ-dense in R, we have that $ann(x + IJM)$ is τ-dense in R. As x was arbitrary we find $ann((IM \cap JM)/(IJ)M) \supseteq I + J$. Thus we have the desired result.

The author can find the following relative Chinese RemainderTheorem in [3]. The version of Porter gave us an impression to study it.

THEOREM 2 (PORTER). *Let R be a commutative ring and τ be a torsion theoryon R-Mod. Suppose that $Max_\tau(R) \neq \phi$ and let $\{I_i | i = 1, 2, \cdots, n\}$ be a finite family of pairwise τ-comaximal idelas in R. For any left R-module M, we have*

(1) $(\Pi_{i=1}^n I_i)M \longrightarrow (\cap_{i=1}^n I_i)M$ is τ-surjective and

(2) $M \longrightarrow \oplus_{i=1}^n M/I_iM$ is τ-surjective with kernel $\cap_{i=1}^n I_iM$

The condition $\text{Max}_\tau(R) \neq \phi$ was used by the fact that every member in $\text{Max}_\tau(R)$ is prime ideal in R, which is Albu and Năstăsescu's work [1].

In order to drop the condition $\text{Max}_\tau(R) \neq \phi$, we need a lemma, which is useful in the proof of main Theorem.

LEMMA 3. *Let R be a commutative ring and $\{I_i | i = 1, 2, \cdots, n\}$ be pairwise τ-comaximal idels of R. Let M be any left R-module, then we have the following :*

(1) $I_i + \cap_{j \neq i} I_j$ is τ-dense in R for each $i = 1, 2, \cdots, n$

(2) $I_iM + (\cap_{j \neq i} I_j)M$ is τ-dense in M for each $i = 1, 2, \cdots, n$.

Proof. (1) We prove for the case $I_1 + D_1$ is τ-dense in R, where $D_1 = \cap_{j \neq 1} I_j$. For the case $n = 1$ is clear. Assume that $I_1 + \cap_{j=2}^k I_j$ is τ-dense in R.

Note that $I_1 + \cap_{j=2}^{k+1} I_j$ contains $(I_1 + \cap_{j=2}^k I_j)(I_1 + I_{k+1})$, which is τ-dense in R, thus $I_1 + \cap_{j=2}^{k+1}$ is τ-dense in R i.e., the induction step is proved. Consequently $I_1 + \cap_{j=2}^n I_j = I_1 + D_1$ is τ-dense in R. A similar argument shows that for each $i = 1, 2, \cdots, n$, $I_i + D_i$ is τ-dense in R, where $D_i = I_1 \cap I_2 \cap \cdots I_{i-1} \cap I_{i+1} \cap \cdots \cap I_n$.

(2) For each $i = 1, 2, \cdots, n$, note that $I_iM + D_iM = (I_i + D_i)M$. $M/(I_i + D_i)M$ can be a left $R/(I_i + D_i)$-module by the action $(r + I_i + D_i)(m + (I_i + D_i)M) = rm + (I_i + D_i)M$

We regard$M/(I_i + D_i)M$ as a homomorphic image of free $R/(I_i + D_i)$-module $\oplus_{\alpha \in M}(R/(I_i + D_i))_\alpha$, by (1) $R/(I_i + D_i)$ is τ-torsion and τ-torsion class is closed under direct sum, we have that $I_iM + D_iM$ is τ-dense in M.

THEOREM 4.(RELATIVE CHINESE REMAINDER THEOREM). *Let R be a commutative ring and $\{I_i | i = 1, 2, \cdots, n\}$ be a finite family of pairwise τ-comaximal ideals in R. For any left R-module M, we have*

(1) $(\Pi_{i=1}^n I_i)M \longrightarrow (\cap_{i=1}^n I_i)M$ is τ-surjective and

(2) $M \longrightarrow \oplus_{i=1}^n M/I_iM$ is τ-surjective with kernel $\cap_{i=1}^n I_iM$

Proof. (1) The case $n = 1$ is trivial. Assume the result holds for any left R-module M and all families of pairwise τ-comaximal ideals having fewer than n. Consider $\{I_i | i = 1, 2, \cdots, n\}$ and we denote by $P_i = \Pi_{j \neq i} I_j$ and $D_i = \cap_{j \neq i} I_j$ We want to show that $I_i + P_i$ is τ-dense in R. By Lemma 3 (1), for each $i = 1, 2, \cdots, n$, I_i and D_i is τ-comaximal ideals in R.Now apply to Lemma 1, we have that $\frac{I_i + D_i}{I_i D_i}$is τ-torsion, so its homomorphic image $\frac{I_i + D_i}{I_i + P_i}$ is τ-torsion. Consider the following short exact sequence,

$$0 \longrightarrow \frac{I_i + D_i}{I_i + P_i} \longrightarrow \frac{R}{I_i + P_i} \longrightarrow \frac{R}{I_i + D_i} \longrightarrow 0$$

By the Lemma 3(1), $R/I_i + D_i$ is τ-torsion module. And the τ-torsion class is closed under extension, so we have $R/(I_i + P_i)$ is τ-torsion, thus $I_i + P_i$ is τ-dense in R.

Now we can apply the Lemma 1, and get

$\left(\Pi_{k=1}^n I_k\right)M = I_iP_iM \longrightarrow I_iM \cap P_iM$ is an τ-epimorphism.

Now by the induction hypothesis, $I_iM \cap P_iM \longrightarrow I_iM \cap (D_iM)$ is τ-surjection.

Thus $(\Pi_{k=1}^n I_k)M \longrightarrow (\cap_{k=1}^n I_k)M$ is τ-surjection.

(2) The case $n = 1$ is clear.

We also assume the result holds for any left R-module M and all families of pairwise τ-comaximal ideals having fewer than n.

Consider the following short exact sequence

$$0 \longrightarrow \frac{M}{(\cap_{j\neq i} I_j M) \cap I_i M} \longrightarrow \frac{M}{D_i M} \oplus \frac{M}{I_i M} \longrightarrow \frac{M}{D_i M + I_i M} \longrightarrow 0$$

By the Lemma 3(2), $M/(D_i M + I_i M)$ is τ-torsion. Thus $M/(D_i M \cap I_i M)$ is τ-dense in $M/D_i M \oplus M/I_i M$. Now apply the induction hypothesis

$$\frac{M}{D_i M} \oplus \frac{M}{I_i M} \longrightarrow \oplus_{j\neq i} \frac{M}{I_j M} \oplus \frac{M}{I_i M} \simeq \oplus_{i=1}^{n} \frac{M}{I_i M}$$

is τ-surjection. Thus we have the desired result.

We examine R-submodules $\{I_i M | i = 1, 2, \cdots, n\}$ of M in above lemmas and theorems, and consider the following concept in module theoretic sense.

DEFINITION. Let M be a left R-module, a set of left$R$-submodules of M $\{N_i | i = 1, 2, \cdots, n\}$ is called *τ-coindependent* in M if (i) each N_i is not τ-dense in M and (ii) $N_i + \cap_{j\neq i} N_j$ is τ-dense in M for each $i = 1, 2, \cdots, n$.

For example, given pairwise τ-commaximal ideals of commutative ring R $\{I_i | i = 1, 2, \cdots, n\}$, consider left R-submodules $\{I_i M | i = 1, 2, \cdots, n\}$, then the Lemma 3(2) shows that $\{I_i M | i = 1, 2, \cdots, n\}$ is a set of τ-coindependent in M.

Properties on τ-coindependent submodules can be found in [4].

In here,we mention only the fact related with Relative Chinese Remainder Theorem.

PROPOSITION 5. *Let R be a ring with identity (R may not be commutative) and let $\{N_i | i = 1, 2, \cdots, n\}$ be a set of τ-coindependent R-submodules of M. Then we have $M \longrightarrow \oplus_{i=1}^{n} \frac{M}{N_i}$ is τ-surjectivewith kernel $\cap_{i=1}^{n} N_i$.*

Proof. The case $n = 1$ is clear. We assume for any left R-module M and all families of τ-coindependent submodules having less than n. Consider the following short exact sequence ;

$$0 \longrightarrow \frac{M}{(\cap_{i=1}^{n-1} N_i) \cap N_n} \longrightarrow \frac{M}{\cap_{i=1}^{n-1} N_i} \oplus \frac{M}{N_n} \longrightarrow \frac{M}{\cap_{i=1}^{n-1} N_i + N_n} \longrightarrow 0$$

By the τ-coindepency of $\{N_i | i = 1, 2, \cdots, n\}$, $\cap_{i=1}^{n-1} N_i + N_n$ is τ-densein M. Use the induction hypothesis we have the result.

COROLLARY 6. *If $Max_\tau(M)$ is finite, then $M/J_\tau(M)$ is τ-semisimple τ-artinian, where $J_\tau(M)$ is the relative Jacobson radical of M.*

Proof. Since $\mathrm{Max}_\tau(M)$ is finite, $J_\tau(M) = \cap_{i=1}^{n} N_i$, where N_i is τ-critical submodules of M. And the set $\{N_i | i = 1, 2, \cdots, n\}$ forms a τ-coindependent submodules in M, then the relative Chinese Remainder Theorem (Theorem 4) gives an τ-epimorphism $\frac{M}{J_\tau(M)} \longrightarrow \oplus_{i=1}^{n} \frac{M}{N_i}$.

Hence $\frac{M}{J_\tau(M)}$ is τ-semisimple and τ-artinian as left R-module.

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