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COMPLETE MAXIMAL SPACE–LIKE HYPERSURFACES IN AN ANTI–DE SITTER SPACE

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1. Introduction

It is well known that there exist no closed minimal surfaces in a 3-dimensional Euclidean space $\mathbb{R}^3$. Myers [4] generalized the result to the case of the higher dimension and proved that there are no closed minimal hypersurfaces in an open hemisphere. The complete and non-compact version concerning Myers' theorem is recently considered by Cheng [1] and the following theorem is proved.

THEOREM. *Let M be a complete minimal hypersurface in an (m+1)-dimensional sphere. If M is contained in the hemisphere and if the volume of M is finite, then it is totally geodesic.*

On the other hand, it is pointed out in a series of papers by Choquet, Fisher and Marsden [2], Marsden and Tipler [3], Stumbles [5] and so on that maximal space-like hypersurfaces or space-like hypersurfaces of constant mean curvature in a Lorentzian space form play an important role in relativity theory. Let R_2^{m+2} be an (m+2)-dimensional indefinite Euclidean space with index 2 whose scalar product is defined by

$$< x, x >= \sum_{j=1}^{m}(x_j)^2 - (x_{m+1})^2 - (x_{m+2})^2,$$

where $x = (x_1, \ldots, x_{m+2})$ in R_2^{m+2}. Let $H_1^{m+1}(c)$ be an (m+1)- dimensional anti-de Sitter space of constant curvature c which is defined by

$$H_1^{m+1}(c) = \{x \in R_2^{m+2} : < x, x >= \frac{1}{c} < 0\}.$$

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The purpose of this note is to deal with the Myers type theorem in an anti-de Sitter space and to prove the following

THEOREM. *Let M be a complete maximal space-like hypersurface in an $(m+1)$-dimensional anti-de Sitter space $H_1^{m+1}(c)$. If the isometric immersion f satisfies*

$$< f, v > \leq a$$

for some constant, where v is a unit time-like vector, then it is totally geodesic.

2. Preliminaries

In this section we recall fundamental properties about space-like submanifolds of an indefinite submanifold. Let $M = M^m$ be an m-dimensional space-like submanifold of an n-dimensional indefinite Riemannian manifold $N = N_r^n$ of index $r > 0$ and let N be the submanifold of an l-dimensional indefinite Riemannian manifold $L = L_s^l$ of index $s \geq r$. Let X and Y be any two vector fields tangent to M and let ∇^M and ∇^N be the Levi-Civita connections of M and N, respectively. Then the Gauss equation shows

$$\nabla_X^N Y = \nabla_X^M Y + \alpha_M(X, Y), \tag{2.1}$$

where α_M is the second fundamental form of M in N. Let ∇^L be the Levi-Civita connection of L and α_N be the second fundamental form of N in L. Then we have

$$\nabla_X^L Y = \nabla_X^N Y + \alpha_N(X, Y), \tag{2.2}$$

for any vector fields tangent to N. From (2.1) and (2.2) we find

$$\nabla_X^L Y = \nabla_X^M Y + \alpha_M(X, Y) + \alpha_N(X, Y) \tag{2.3}$$

for any vector fields X and Y tangent to M. Since M can be regarded as the submanifold of L, the equation (2.3) shows that the second fundamental form α of M in L is given by

$$\alpha(X, Y) = \alpha_M(X, Y) + \alpha_N(X, Y), \tag{2.4}$$

where $\alpha_M(X, Y)$ is tangent to N and $\alpha_N(X, Y)$ is normal to N. We denote by $\mathbf{h}$ and $\mathbf{h}_M$ the mean curvature vector fields of M in L and N, respectively. Then we have

$$\mathbf{h} = \mathbf{h}_M + \mathbf{h}_N(M), \tag{2.5}$$

where $\mathbf{h}_N(M)$ is the vector field normal to N given by

$$\mathbf{h}_N(M) = \frac{1}{m} \sum_{i=1}^{m} \alpha_N(E_i, E_i),$$

where $\{E_1, \ldots, E_m\}$ is the local field of orthonormal frames adapted to the Riemannian metric in M, which is called the *relative mean curvature vector field* of M with respect to N and L. The submanifold M in N (resp. in L) is said to be *maximal* if $\mathbf{h}_M$ (resp. $\mathbf{h}$) vanishes identically. From (2.5) we have

LEMMA 2.1. *The space-like submanifold M in N is maximal if and only if the mean curvature vector field $\mathbf{h}$ of M in L is normal to N.*

LEMMA 2.2. *The space-like submanifold M in L is maximal if and only if M is maximal in N and the relative mean curvature vector field $\mathbf{h}_N(M)$ of M with respect to N and L vanishes identically.*

The following generalized maximum principle due to Omori [5] and Yau [7] will play an important role in this paper.

THEOREM 2.3. *Let M be an m-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M, then for any $\varepsilon > 0$, there exists a point p in M such that*

$$F(p) + \varepsilon > \sup F, \qquad |\mathrm{grad} F|(p) < \varepsilon, \qquad \triangle F(p) < \varepsilon.$$

3. Space-like submanifolds

This section is concerned with the space-like submanifold in an indefinite Euclidean space. Let $\mathbf{R}_{q+1}^{m+q+1}$ be an $(m+q+1)$-dimensional indefinite Euclidean space of index $(q+1)$ whose scalar product $<,>$ is given by

$$<x,y>=\sum_{i=1}^{m} x_i y_i - \sum_{\alpha=m+1}^{m+q+1} x_\alpha y_\alpha$$

for any point $x=(x_1,\ldots,x_{m+q+1})$ and $y=(y_1,\ldots,y_{m+q+1})$ in $\mathbf{R}_{q+1}^{m+q+1}$ and let $N=(H_q^{m+q}(c),h)$ be an $(m+q)$-dimensional pseudohyperbolic space of constant curvature c with indefinite metric h. It is defined by

$$H_q^{m+q}(c)=\left\{x\in R_{q+1}^{m+q+1} : <x,x>=-r^2=\frac{1}{c}, r>0\right\}.$$

Let (M,g) be a space-like submanifold of $H_q^{m+q}(c)$. The submanifold M is said to be *full* if there exist no totally geodesic hypersurfaces $H_{q-1}^{m+q-1}(c)$ in $H_q^{m+q}(c)$ which contain M. Then we first prove the following

THEOREM 3.1. *Let M be an m-dimensional complete space-like maximal submanifold of $H_q^{m+q}(c)$. If the isometric immersion f satisfies*

$$|<f,v>|\leq a \tag{3.1}$$

for some positive constant a, where v is a unit time-like vector, then M is not full.

Proof. By an isometry on the indefinite Euclidean space $\mathbf{R}_{q+1}^{m+q+1}$ the time-like vector v translates a vector $e_{m+2}=(0,\ldots,0,1)$. For any nonnegative constant b the subset ${}_bH_q^{m+q}(c)$ of $H_q^{m+q}(c)$ is defined by

$${}_bH_q^{m+q}(c)=\left\{x\in H_q^{m+q}(c) : |x_{m+q+1}|\leq b\right\}.$$

Then the condition (3.1) is equivalent to the fact that M is contained in ${}_aH_q^{m+q}(c)$.

Since the pseudohyperbolic space $N = H_q^{m+q}(c)$ is totally umbilic in $\mathbf{R}_{q+1}^{m+q+1}$, the second fundamental form α_N of $H_q^{m+q}(c)$ in $\mathbf{R}_{q+1}^{m+q+1}$ is expressed as

$$\alpha_N(X,Y) = h(X,Y)\xi_N \tag{3.2}$$

for any vector fields X and Y tangent to $H_q^{m+q}(c)$, where ξ_N is called the *normal curvature vector field* on $H_q^{m+q}(c)$. To each point p in $H_q^{m+q}(c)$ the corresponding position vector field is denoted by P. Then P/r is the unit normal to $H_q^{m+q}(c)$ outward away from the origin, and it follows that the normal curvature vector field is given by

$$\xi_N = \frac{1}{r^2}P = -cP \tag{3.3}$$

because the sign of the hypersurface $N = H_q^{m+q}(c)$ in $\mathbf{R}_{q+1}^{m+q+1}$ is equal to 1, that is, the coindex of N is 1. Thus we have

$$\alpha_N(X,Y) = -ch(X,Y)P \tag{3.4}$$

by (3.2) and (3.3). When M is regarded as the submanifold in $\mathbf{R}_{q+1}^{m+q+1}$, let α be the second fundamental form of M in $\mathbf{R}_{q+1}^{m+q+1}$. By (2.4) α is given by

$$\alpha(X,Y) = \alpha_M(X,Y) - ch(X,Y)P$$

for any vector fields X and Y tangent to M. This implies that the mean curvature vector field $\mathbf{h}_M$ and $\mathbf{h}$ of M in $H_q^{m+q}(c)$ and $\mathbf{R}_{q+1}^{m+q+1}$ satisfy

$$\mathbf{h} = \mathbf{h}_M - cP. \tag{3.6}$$

On the other hand, since we can consider any point p in M as the one in $\mathbf{R}_{q+1}^{m+q+1}$, we put $p = (p_i, p_\alpha) = (p_A)$, where the following convention on the range of indices is used, unless otherwise stated :

$$1 \leq A, B, \cdots \leq m+q+1, \quad 1 \leq j, k, \cdots \leq m, \quad m+1 \leq \alpha \leq m+q+1.$$

Then the position vector field P is expressed as $P = \sum p_A \partial_A$ in terms of the local coordinate system $\{x_A\}$, where $\partial_A = \partial/\partial x_A$. For any vector field Y tangent to M we have

$$Y_p = Y. \tag{3.7}$$

In fact, we can consider $Y_p = (Y_{PA})$ as the $\mathbf{R}_{q+1}^{m+q+1}$-vector field on M, that is, the vector field on $\mathbf{R}_{q+1}^{m+q+1}$ along M. Then it coincides with the vector field $\nabla_Y^L P$ because the ambient space is an indefinite Euclidean space, where ∇^L is the Levi-Civita connection on $L = \mathbf{R}_{q+1}^{m+q+1}$. Thus we get

$$XY_P = \nabla_X^L(Y_P) = \nabla_X^L Y.$$

Accordingly, by (3.7) the Laplacian $\triangle_M$ of p on the space-like submanifold M is given as

$$\begin{aligned}\triangle_M p &= \sum (E_k E_k p - \nabla_{E_k}^M E_k p)\\ &= \sum (\nabla_{E_k}^L E_k - \nabla_{E_k}^M E_k)\\ &= \sum \alpha(E_k, E_k),\end{aligned}$$

where $\{E_k\}$ is a local field of orthonormal frames adapted to the Riemannian metric g on M. Thus we get

$$\triangle_M p = m\mathbf{h},$$

which together with (3.6) implies that

$$\triangle_M p = m(\mathbf{h}_M - cP) \tag{3.8}$$

From now on we assume that the space-like submanifold M is maximal in $H_q^{m+q}(c)$ in $\mathbf{R}_{q+1}^{m+q+1}$. By (3.8) the $(m+q+1)$-component x_{m+q+1} for any point $x = (x_A)$ in M satisfies

$$\triangle_M x_{m+q+1} = -cmx_{m+q+1}. \tag{3.9}$$

Since the Ricci curvature of M is bounded from below by a constant $m(m-1)c$ and the function $f = x_{m+q+1}$ is bounded from above by the assumption, we can apply Generalized Maximal Principle to the function f. For any sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0 (n \to \infty)$, there exists a sequence $\{x_n\}$ of points in M such that

$$(3.10)\quad f(x_n) + \varepsilon_n > \sup f, \qquad |\operatorname{grad} f|(x_n) < \varepsilon_n, \qquad \triangle f(x_n) < \varepsilon_n.$$

By (3.9) and (3.10) we get

$$\lim_{n\to\infty} f(x_n) = \sup f \le 0$$

because c is negative. On the other hand, since f is also bounded from below, we can apply Theorem 2.3 to the function $-f$, it is similarly seen that for the same sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0(n \to \infty)$, there exists a sequence $\{y_n\}$ of points in M such that

$$f(y_n) - \varepsilon_n < \inf f, \qquad |\operatorname{grad} f|(y_n) < \varepsilon_n, \qquad \triangle f(y_n) > -\varepsilon_n,$$

which implies

$$\lim_{n\to\infty} f(p_n) = \inf f \ge 0.$$

Thus we get $f \equiv 0$ on M, which yields that M is contained in the totally geodesic hypersurface $H^{m+q-1}_{q-1}(c)$ in $H^{m+q}_q(c)$ defined by $x_{m+q+1} = 0$. It completes the proof.

Theorem in the introduction is easily verified from Theorem 3.1.

REMARK. An $(m + q + 1)$-dimensional indefinite Euclidean space $\mathbf{R}^{m+q+1}_{q+1}$ of index $q + 1$ can be first regarded as a product manifold of

$$\mathbf{R}^{m_1+1}_1 \times \cdots \times \mathbf{R}^{m_{q+1}+1}_1,$$

where $\sum_{r=1}^{q+1} m_r = m$. With respect to the standard orthonormal basis of $\mathbf{R}^{m+q+1}_{q+1}$ a class of space-like submanifolds

$$H^{m_1}(c_1) \times \cdots \times H^{m_{q+1}}(c_{q+1})$$

of $\mathbf{R}^{m+q+1}_{q+1}$ is defined as the pythagorean product

$$H^{m_1}(c_1)\times\cdots\times H^{m_{q+1}}(c_{q+1})$$
$$=\Big\{(x_1,\ldots,x_{q+1})\in\mathbf{R}^{m+q+1}_{q+1}=\mathbf{R}^{m_1+1}_1\times\cdots\times\mathbf{R}^{m_{q+1}+1}_1 :$$
$$|x_r|^2=-\frac{1}{c_r}>0\Big\},$$

where $r=1,\ldots,q+1$. The mean curvature vector $\mathbf{h}$ of M is given by

$$\mathbf{h}=\frac{1}{m}\sum_{r=1}^{q+1}(m_r c_r x_r)-cx \tag{3.12}$$

at $x=(x_1,\ldots,x_{q+1})\in M$, which is parallel in the normal bundle of M. This means that M is maximal if it satisfies $n_r c_r = nc$ for any index r by (3.12). It gives us that the assumption that M is contained in ${}_aH^{m+q}_q(c)$ is necessary.

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