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SOME BOUNDS FOR ASSIGNMENTS

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1. Introduction

Let $A = [a_{ij}]$ be an $m \times n$ matrix over any commutative ring, $m \leq n$. The *permanent* of A, written Per (A) or simply Per A, is defined by

$$\text{Per } A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{m\sigma(m)} \tag{1.1}$$

where the summation extends over all one-to-one functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. The special case $m = n$ is of particular important. We denote the permanent of a square matrix A by per A instead of Per A.

R. A. Brualdi, D. J. Hartfiel and S. G. Hwang [3] introduced a class of function generalizing the permanent function and which, like the permanent, are combinatorially significant as counting functions.

Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots s_n)$ be positive integral vectors satisfying $r_1 + \cdots + r_m = s_1 + \cdots + s_n$, and let $\mathcal{U}(R, S)$ denote the class of all $m \times n$ matrices $A = [a_{ij}]$ of 0's and 1's such that

$$\sum_{k=1}^{n} a_{ik} = r_i, \ \sum_{k=1}^{m} a_{kj} = s_j, \ \ i = 1, \cdots, m, \ \ j = 1, \cdots, n. \tag{1.2}$$

Thus R is the *row sum vector* and S is the *column sum vector* of every matrix in $\mathcal{U}(R, S)$. We assume thoughout that $\mathcal{U}(R, S) \neq \phi$. We refer to matrices in $\mathcal{U}(R, S)$ as (R, S)-*assignments* or as *assignments* when R and S are fixed in the discussion.

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Let $X = [x_{ij}]$ be an $m \times n$ matrix. We define the *support* of $A = [a_{ij}]$ to be the set $supp(A) = \{(i,j) : a_{ij} \neq 0\}$ The $(R,S)-$ *assignment function* $\mathrm{P}_{R,S}(\cdot)$ is now defined by

$$(1.3) \qquad \mathrm{P}_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \mathrm{supp}(A)} x_{ij}.$$

In case X is a $(0,1)$-matrix, then

$$(1.4) \qquad \mathrm{P}_{R,S}(X) = |\{A \in \mathcal{U}(R,S) : A \leq X\}|,$$

where $A \leq X$ means that no entry of A exceeds the corresponding entry of X. Thus $\mathrm{P}_{R,S}(X)$ counts the number of matrices of $\mathcal{U}(R,S)$ which are less than or equal to X.

Now let $\overline{\mathcal{U}(R,S)}$ denote the convex hull of the (R,S)-assignments considered as points in a mn-dimensional real space. Because each assignment has all entries equal to 0 or 1, it follows readily that the assignments are precisely the vertices(extreme points) of $\overline{\mathcal{U}(R,S)}$. Brualdi, Hartfiel and Hwang [3] proved that $\overline{\mathcal{U}(R,S)}$ is a convex polytope.

For integers k, n, $1 \leq k \leq n$, let $V_{k,n}$ denote the set of all $n \times 1$ (0,1)-matrices whose entries have sum k. For real n-vectors, i.e., real $n \times 1$ matrices $\mathbf{x}$ and $\mathbf{y}$ we say that $\mathbf{x}$ *is majorized by* $\mathbf{y}$ (or $\mathbf{y}$ *majorizes* $\mathbf{x}$), written as $\mathbf{x} \prec \mathbf{y}$ if

$$(1.5) \qquad \max\{\mathbf{v}^T\mathbf{x} : \mathbf{v} \in V_{k,n}\} \leq \max\{\mathbf{v}^T\mathbf{y} : \mathbf{v} \in V_{k,n}\}$$

for all $k = 1, \cdots, n$ and equality holds in (1.5) when $k = n$. $\mathbf{x}$ is said to be *submajorized* by $\mathbf{y}$, written as $\mathbf{x} \prec_w \mathbf{y}$, if (1.5) holds for all $k = 1, \cdots, n$. It is well known that if $\mathbf{x}$ is majorized by $\mathbf{y}$ then, for all convex function φ, $(\varphi(x_1), \cdots, \varphi(x_n))^T$ is submajorized by $(\varphi(y_1), \cdots, \varphi(y_n))^T$.

2. The Bounds for $\mathbf{P}_{R,S}(\cdot)$ in $\overline{\mathcal{U}(R,S)}$

LEMMA 2.1 [5]. *Let $R = (r_1, \cdots r_n)$ be a positive integral n-vectors. Then, for any $A \in \overline{\mathcal{U}(R)}$,*

$$\text{per } A \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}. \tag{2.1}$$

Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be positive integral vectors and let $X = [x_{ij}]$ be an $m \times n$ matrix. Let X_{is_i} be the $m \times s_i$ matrix each of whose columns equals to column i of X, $i = 1, \cdots, n$. Let t be $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Let Y be the $m \times (mn - t)$ matrix given by

$$Y = [\,Y_1 \quad Y_2 \quad \cdots \quad Y_m\,], \tag{2.2}$$

where Y_i, $i = 1, \cdots m$, are matrices which ith row entries are all 1 otherwise 0. That is,

$$Y_i = \begin{bmatrix} 0 & \cdots & 0 \\ & \vdots & \\ 1 & \cdots & 1 \\ & \vdots & \\ 0 & \cdots & 0 \end{bmatrix}_{m \times (n - r_i)}.$$

Finally, let Z be the $mn \times mn$ matrix defined by

$$Z = \begin{bmatrix} X_{1s_1} & & & Y \\ & X_{2s_2} & 0 & Y \\ & 0 & \ddots & \vdots \\ & & X_{ns_n} & Y \end{bmatrix}. \tag{2.3}$$

LEMMA 2.2 [3]. *Let R, S and Z be as given in the preceding statements. Then*

$$P_{R,S}(X) = \frac{\text{per } Z}{\prod_{i=1}^{m} (n - r_i)! \prod_{j=1}^{n} s_j!}. \tag{2.4}$$

THEOREM 2.1. *Let $u = |\mathcal{U}(R,S)|$. Then, for any $X \in \overline{\mathcal{U}(R,S)}$,*

(2.5)
$$\left(\frac{1}{u}\right)^{t-1} \le P_{R,S}(X) \le \min\left\{ \frac{(n!)^{m-\frac{t}{n}}}{\prod_{i=1}^{m}(n-r_i)!}, \ \frac{(m!)^{n-\frac{t}{m}}}{\prod_{j=1}^{n}(m-s_j)!} \right\}.$$

Proof. If $X \in \overline{\mathcal{U}(R,S)}$, then $X = \lambda_1 A_1 + \cdots + \lambda_u A_u$ where $\sum_{i=1}^{u} \lambda_i = 1$, $0 \le \lambda_i \le 1$, and each $A_i \in \mathcal{U}(R,S)$, $i = 1, \cdots u$. Thus

$$\begin{aligned} \mathrm{P}_{R,S}(X) &= \mathrm{P}_{R,S}(\lambda_1 A_1 + \cdots + \lambda_u A_u) \\ &= \sum_{A_i \in \mathcal{U}(R,S)} \prod_{(k,l)\in \mathrm{supp}(A_i)} x_{kl} \\ &\ge \lambda_1^t + \lambda_2^t + \cdots + \lambda_u^t. \end{aligned}$$

Let $\varphi(x) = x^t$. Then $\varphi(x)$ is an increasing convex function on $[0,1]$. Without loss of generality, we may assume $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_u$. Since $(\frac{1}{u}, \cdots, \frac{1}{u}) \prec (\lambda_1, \cdots, \lambda_u)$, $((\frac{1}{u})^t, \cdots, (\frac{1}{u})^t) \prec_w (\lambda_1^t, \cdots, \lambda_u^t)$. Therefore,

$$\begin{aligned} \mathrm{P}_{R,S}(X) &\ge \lambda_1^t + \cdots + \lambda_u^t \\ &\ge (\frac{1}{u})^t + \cdots + (\frac{1}{u})^t \\ &= (\frac{1}{u})^{t-1}. \end{aligned} \tag{2.6}$$

Next, we prove the upper bound for $\mathrm{P}_{R,S}(X)$, $X \in \overline{\mathcal{U}(R,S)}$. By (2.1) and (2.4),

$$\mathrm{P}_{R,S}(X) \le \frac{\prod_{i=1}^{mn} (z_i!)^{\frac{1}{z_i}}}{\prod_{i=1}^{m}(n-r_i)! \prod_{j=1}^{n} s_j!},$$

where z_i is the sum of all entries in the ith column of Z, $i = 1, \cdots, mn$. Then

$$(z_i)^{\frac{1}{z_i}} = \begin{cases} (s_1)!^{\frac{1}{s_1}}, & i = 1, \cdots, s_1; \\ (s_2!)^{\frac{1}{s_2}}, & i = s_1 + 1, \cdots, s_1 + s_2; \\ \vdots & \\ (s_n!)^{\frac{1}{s_n}}, & i = t - s_n + 1, \cdots, t; \\ (n!)^{\frac{1}{n}}, & i = t+1, \cdots, mn. \end{cases}$$

Thus, we have

$$\mathrm{P}_{R,S}(X) \leq \frac{1}{\prod_{i=1}^{m}(n-r_i)!\prod_{j=1}^{n}s_j!}\{(s_1!)^{\frac{1}{s_1}}\}^{s_1}\cdots\{(s_n!)^{\frac{1}{s_n}}\}^{s_n}\{(n!)^{\frac{1}{n}}\}^{mn-t}$$

$$(2.7) \qquad = \frac{(n!)^{m-\frac{t}{n}}}{\prod_{i=1}^{m}(n-r_i)!}.$$

Similarly, we can obtain that

$$(2.8) \qquad \mathrm{P}_{R,S}(X) \leq \frac{(m!)^{n-\frac{t}{m}}}{\prod_{j=1}^{n}(m-s_j)!},$$

and, by (2.6), (2.7) and (2.8), the proof is completed.

Let $A=[a_{ij}]$ be an $m\times n$ real matrix with row vectors $\alpha_1,\alpha_2,\cdots,\alpha_m$. We say A is *contractible on column*(resp. *row*) k if column (resp. row) k contains exactly two nonzero entries. Suppoe A is contractible on column k with $a_{ik}\neq 0\neq a_{jk}$ and $i\neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i+a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki}\neq 0\neq a_{kj}$ and $i\neq j$, then the matrix $A_{k:ij}=[A_{ij:k}]^T$ is called the contraction of A on row k relative to column i and j. We say that A can be *contracted* to a matrix B if either $B=A$ or there exist matrices $A_0,A_1,\cdots,A_t$ $(t\geq 1)$ such that $A_0=A$, $A_t=B$ and A_r is a contraction of A_{r-1} for $r=1,\cdots,t$.

LEMMA 2.3 [1]. *Let A be a nonnegative real matrix of order $n>1$ and let B be a contraction of A. Then*

$$(2.9) \qquad \mathit{per}\, A = \mathit{per}\, B.$$

LEMMA 2.4 [3]. *Let $R=(1,\cdots,1)$ be the m-tuple of $1's$, and let $S=(s_1,s_2,\cdots,s_n)$ where the s_j are positive integers with $s_1+\cdots+s_n=m$. Let $X=[x_{ij}]$ be an $m\times n$ real matrix. For each $j=1,\cdots,m$, let X_j be the matrix $(\frac{1}{s_j})X_{j,s_j}$ where X_{j,s_j} is the $m\times s_j$ matrix each* *of whose columns equals column j of X.Finally, let $\widetilde{X} = [X_1, \cdots, X_n]$. Then*

$$P_{R,S}(X) = \left(\prod_{k=1}^{n} \frac{s_k^{s_k}}{s_k!} \right) \operatorname{per} \widetilde{X}. \tag{2.10}$$

LEMMA 2.5. *Let $R = kE_n$ where k is an positive integer ($k \leq n$). Let $X = [x_{ij}]$ be an $m \times n$ real matrix. Then*

$$P_{R,S}(X) = P_{R,S}(PX), \tag{2.11}$$

where P is any $m \times m$ permutation matrix.

Proof. For each $A \in \mathcal{U}(R, S)$, $PA \in \mathcal{U}(R, S)$ for any $m \times m$ permutation matrix P.

$$\begin{aligned} \mathrm{P}_{R,S}(X) &= \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \operatorname{supp}(A)} x_{ij} \\ &= \sum_{PA \in \mathcal{U}(R,S)} \prod_{(i',j) \in \operatorname{supp}(PA)} x_{i'j} \\ &= \mathrm{P}_{R,S}(PX). \end{aligned}$$

THEOREM 2.2. *Let $R = (1, \cdots, 1)$ be the m-tuple of $1'$s, and let $S = (s_1, s_2, \cdots, s_n)$ where the s_j are positive integers with $s_1 + \cdots + s_n = m$. Let $X = [x_{ij}]$ be an $m \times n$ nonnegative real matrix which is contractible on the column k. Let Y be the $(m-1) \times (n-1)$ matrix which is a contraction of X. Then*

$$P_{R,S}(X) \geq P_{R',S'}(Y), \tag{2.12}$$

where $R' = (1, \cdots, 1)$ is the $(m-1)$-tuple of $1'$s and $S' = S(\cdot|k)$ with equality if and only if $s_k = 1$.

Proof. Without loss of generality, we may assume that $x_{1k} \neq 0 \neq x_{2k}$ and $x_{ik} = 0$, $i = 3, 4, \cdots, m$.

Case 1. Suppose $s_k = 1$. Since, by (2.9). per $\widetilde{X} =$ per $\widetilde{X}'$ where $\widetilde{X}'$ is the contraction on column $s_1 + \cdots + s_{k-1} + 1$ of $\widetilde{X}$, by (2.10),

$$
\begin{aligned}
\mathrm{P}_{R,S}(X) &= \left(\prod_{j=1}^{n} \frac{s_j^{s_j}}{s_j!} \right) \text{per } \widetilde{X} \\
&= \left(\prod_{j=1}^{k-1} \frac{s_j^{s_j}}{s_j!} \prod_{j=k+1}^{n} \frac{s_j^{s_j}}{s_j!} \right) \text{per } \widetilde{X}'. \qquad (2.13)
\end{aligned}
$$

Let $\widetilde{Y} = [Y_1, \cdots, Y_{n-1}]$ where Y_i is defined the same way as X_i. is, then $\widetilde{X}' = \widetilde{Y}$. Therefore,

$$
\begin{aligned}
\mathrm{P}_{R,S}(X) &= \left(\prod_{j=1}^{k-1} \frac{s_j^{s_j}}{s_j!} \prod_{j=k+1}^{n} \frac{s_j^{s_j}}{s_j!} \right) \text{per } \widetilde{Y} \\
&= \mathrm{P}_{R',S'}(\widetilde{Y}).
\end{aligned}
$$

Case 2. Suppose $s_k = 2$. Let $s_1 + \cdots + s_{k-1} = \sigma_{k-1}$. Then

$$
\begin{aligned}
\mathrm{P}_{R,S}(X) &= \left(\prod_{j=1}^{n} \frac{s_j^{s_j}}{s_j!} \right) \text{per } \widetilde{X} \\
&= \left(\prod_{j=1}^{n} \frac{s_j^{s_j}}{s_j!} \right) \frac{x_{1k}x_{2k}}{2} \text{per } \widetilde{X}(1, 2 \, \sigma_{k-1} + 1, \sigma_{k-1} + 2) \\
&\geq 0.
\end{aligned}
$$

But $\mathrm{P}_{R',S'}(Y) = 0$ because $s_1 + \cdots + s_{k-1} + s_{k+1} + \cdots + s_n \neq m - 1$.

Case 3. Suppose $s_k \geq 3$. Then $\mathrm{P}_{R,S}(X) = 0$ and $\mathrm{P}_{R',S'}(Y) = 0$ because $\sigma_{k-1} + s_{k+1} + \cdots + s_n \neq m - 1$, which completes the proof.

3.The Number of k-factors of Complete Bipartite Graph

Let G be a graph , and let $K_{n,n}$ be a complete bipartite graph. A *factor* of G is a spanning subgraph of G which is not totally disconnected. For a positive integer k, a *k-factor* ia a regular factor of degree k.

Let Λ_n^k denote the set of n-square (0,1)-matrices with k 1's in each row and each column. If $R = S = kE_n$, where E_n is the n-vectors of 1's, then $\mathcal{U}(R, S) = \Lambda_n^k$.

LEMMA 3.1 [6]. *$A \in \Lambda_n^k$, then*

$$A = \sum_{j=1}^{k} P_j, \tag{3.1}$$

where the P_j's are permutation matrices.

Let J be the $n \times n$ matrix all of whose entries are 1. Let $D_n = J - I_n$, where I_n is the identity matrix of order n. The D_n is called the *derangement matrix.* Let $d_n :=$ per D_n, then d_n is the number of the derangements of n elements and

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{3.2}$$

THEOREM 3.1. *The number of 2-factors of $K_{n,n}$ is*

$$\frac{n! d_n}{2}. \tag{3.3}$$

Proof. Since the number of 2-factors of $K_{n,n}$ equals to $|\mathcal{U}(R, S)|$ with $R = S = 2E_n$. By (3.1), for any $A \in \mathcal{U}(R, S)$, $A = P_1 + P_2$ where P_1 and P_2 are permutation matrices which do not overlap. First we can fix P_1 among $n!$ permutation matrices. Without loss of generality, we may assume that P_1 is the identity matrix I_n. Then, the number of possible choices of P_2 equals the permanent value of the derangement matrix D_n.

And then, we also can fix P_2 and choose P_1, i.e., we have the same process for a fixed P_1. Therefore,

$$|\mathcal{U}(R,S)| = \frac{n!d_n}{2}.$$

$R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ are positive integral vectors. If $R' = nE_m - R$ and $S' = mE_n - S$, then $|\mathcal{U}(R,S)| = |\mathcal{U}(R',S')|$ because there is a bijection φ from $\mathcal{U}(R,S)$ to $\mathcal{U}(R',S')$.

Let

$$J_{R,S} = \frac{1}{|\mathcal{U}(R,S)|} \sum_{A \in \mathcal{U}(R,S)} A. \tag{3.4}$$

The vertices of $\overline{\mathcal{U}(R,S)}$ are precisely the matrices in $\mathcal{U}(R,S)$ and hence $J_{R,S}$ is the barycenter of $\overline{\mathcal{U}(R,S)}$.

Corollary 3.2. *If $R = S = (n-2)E_n$, then*

$$\lim_{n \to \infty} P_{R,S}(J_{R,S}) = \infty. \tag{3.5}$$

Proof. Since $R = S = 2E_n$ and $R' = S' = (n-2)E_n$, $|\mathcal{U}(R,S)| = |\mathcal{U}(R',S')|$ and

$$\mathrm{P}_{R,S}(J_{R,S}) = \frac{n!d_n}{2}\left(\frac{n-2}{n}\right)^{n(n-2)}.$$

But $\lim_{n\to\infty} n!(n-2/n)^{n(n-2)} = \infty$. The proof is completed.

In [4], Gibson gave a disproof for the case $k = n-1$ of Brualdi's conjecture [3]. The equation (3.5) gives another disproof for the conjecture.

Let γ be an integer with $0 \le \gamma \le n^2$. Let $\mathcal{V}(n,\gamma)$ be the set of all (0,1)-matrices of order n with exactly γ 0's.

LEMMA 3.2 [2]. *Let A be a matrix in $\mathcal{V}(n,\gamma)$, where $\gamma \leq n^2 - n$. Let σ be the number of 1's in A so that $\sigma = n^2 - \gamma$. Then*

$$\text{per } A \leq (r!)^{(nr+n-\sigma)/r}(r+1)!^{(\sigma-nr)/(r+1)}, \tag{3.6}$$

where $r = \left[\frac{\sigma}{n}\right]$.

THEOREM 3.3. *Let $R = (r_1, \cdots, r_n)$ and $S = (s_1, \cdots, s_n)$ be positive integral vectors such that $\sum_{i=1}^n r_i = \sum_{j=1}^n s_j$.*

$$|\mathcal{U}(R,S)| \leq \frac{(n!)^n}{\prod_{i=1}^n [(n-r_i)!s_i!]}. \tag{3.7}$$

In particular, the equality hold for $R = S = nE_n$.

Proof. Let $\sum_{i=1}^n r_i = \sum_{j=1}^n s_j = t$.

$$\begin{aligned} |\mathcal{U}(R,S)| &= \mathrm{P}_{R,S}(J) \\ &= |\{A \in \mathcal{U}(R,S) : A \leq J\}| \\ &= \frac{\text{per } Z}{\prod_{i=1}^n [(n-r_i)!s_i!]}, \end{aligned}$$

where J is the $n \times n$ matrix all of whose entries are 1, and Z is the $n^2 \times n^2$ matrix as defined as same as (2.3). That is

$$Z = \begin{bmatrix} J_{1s_1} & & & Y \\ & \ddots & 0 & \vdots \\ & 0 & J_{ns_n} & Y \end{bmatrix},$$

where J_{is_i} is a $n \times s_i$ matrix all of whose entries are 1. Since $\sigma(Z) = n^3$, $\gamma \leq n^4 - n^2$ and $r = n$, by (3.6),

$$\text{per } Z \leq (n!)^n.$$

Thus, the proof is completed.

W. D. Wei [7] said that if $R = S = kE_n$ then

$$\frac{n!^k}{k!^n} \leq |\mathcal{U}(R,S)|.$$

COROLLARY 3.4. *Let $R = S = kE_n$, $1 \leq k \leq n$. Then the bounds for the number of k-factors of $K_{n,n}$ is*

$$\frac{n!^k}{k!^n} \leq |\mathcal{U}(R,S)| \leq \binom{n}{k}^n$$

with equality if and only if $k = n$.

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