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STRONG UNIQUE CONTINUATION OF THE SCHRÖDINGER OPERATOR

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1. Introduction

It is well known that if $P(x,D)$ is an elliptic differential operator, with real analytic coefficients, and $P(x,D)u = 0$ in an open, connected subset $\Omega \in R^n$, then u is real analytic in Ω. Hence, if there exists $x_0 \in \Omega$ such that u vanishes of ∞ order at x_0, u must be identically 0. If a differential operator $P(x,D)$ has the above property, we say that $P(x,D)$ has the strong unique continuation property (s.u.c.p.). If, on the other hand, $P(x,D)u = 0$ in Ω, and $u = 0$ in Ω', an open subset of Ω, implies that $u = 0$ in Ω we say that $P(x,D)$ has the unique continuation property (u.c.p.). Finally, if $P(x,D)u = 0$ in Ω, and $\operatorname{supp} u \subset K \subset \Omega$ implies that $u = 0$ in Ω, we sat that $P(x,D)$ has the weak unique continuation property(w.u.c.p.).

The first results in this direction are to be found in the work of T.Carleman[2] in 1939. He was able to show that $P(x,D) = \Delta + V(x)$ in R^2 has the (s.u.c.p.) whenever the function $V(x)$ is in $L^{\infty}_{loc}(R^2)$. In order to prove this result he introduced a method, the so called Carleman estimates, which has permeated almost all subsequent work in this subject. In this context, a Carleman estimate is, roughly speaking, an inequality of the form:

$$||e^{t\phi} f||_{L^2(U)} \leq C||e^{t\phi}\Delta f||_{L^2(U)}$$

for all $f \in C_0^{\infty}(U)$, U an open subset of R^2, and a suitable function ϕ, where the constant C is independent of t, for a sequence of real values of t tending to ∞. There are many papers on which this work is based. For a reference, see the survey by C.E.Kenig[6].

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THEOREM. *Let U be a non-empty connected open subset of R^n, and u be a solution of the differential equation*

$$(\Delta + \nabla W)u = 0 \tag{1}$$

Here Δ is the Laplace operator, $W \in L^r_{loc}(R^n)$ for some suitable r. *If u vanishes at an open subset of u, then $u = 0$ identically on U.*

This kind of unique continuation theorem for the Schrödinger operator $\Delta + V$ when $V \in L^{n/2}(R^n)$ has been studied by many mathematicians.([2][3]][4][5])

In [4] D. Jerison showed (s.u.c.p.) holds for the opertor $\Delta + V$, where $V \in L^{n/2}_{loc}(R^n)$. He also suggested unique continuation hold for operators of the form $\Delta + V$, where $V = \sum \partial V_j/\partial x_j$, and $V_j \in L^r_{loc}(R^n)$, $r = (3n-2)/2$.

This last hypothesis on the potential is closer in spirit to the condition on potentials advanced by B.Simon[8].

In [7], unique continuation of the differential equation

$$(\Delta + \sum a_j\partial/\partial x_j + b)u = 0 \tag{2}$$

with $a_j \in L^r_{loc}(R^n)$, $b \in L^r_{loc}(R^n)$ was shown.

To prove unique continuation for (1), we need the following Carleman inequality proved by the author[4].

$$||e^{ts(y)}\nabla f||_{L^{q_1}(U\setminus\{0\})} \leq C||e^{ts(y)}\Delta f||_{L^p(U\setminus\{0\})}. \tag{3}$$

where $1/p - 1/q_1 = 1/r$ for C independent of t as t goes to infinity, $f \in C_0^\infty(U \setminus \{0\})$, and $s(y)$ is a suitable weight function which is radial and radially decreasing. The key feature that distinguishes this inequality from ordinary Sobolev inequalities is that the constant C is independent of the parameter t.

Since these are Sobolev inequalities, exponent $s = n/2$ is the natural one we can expect. But we need some restriction for r. In our case the largest value we can expect for r is $r = (3n-2)/2$, and we will choose $p = (6n-4)/(3n-2)$. Then $q_1 = 2$, $q_2 = (6n-4)/(3n-6)$.

Furthermore, we will use the weight function $s(y)$ defined implicilty by $y = -s(y) + e^{-\epsilon s(y)}$ when $y = log|x| < 0$. Roughly speaking, $e^{ts(y)} \sim |x|^{-t}$. The idea is from Alinhac - Baouendi [1] and has been used by Hörmande[3].

Notations

1.The Diracoperator is a first-order constant coefficient operator on R^n of the form $D = \sum_{j=1}^{n} \alpha_j \partial/\partial x_j$, where $\alpha_1, ...\alpha_n$ are skew hermitian matrices satisfying the Clifford relations : $\alpha_j^* = -\alpha_j$ and $\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk}$; $j,k = 1,...,n$. Also $D^2 = -\Delta$.

2.Polar coordinates

Let S denote the unit sphere in R^n.For$y \in R$, and $w \in S, x = e^y w$ gives polar coordinates on R^n, i.e., $y = \log|x|$ and $w = x/|x|$.The operator

$$L = \sum_{j<k} \alpha_j\alpha_k(x_j\partial/\partial x_k - x_k\partial/\partial x_j)$$

acts only in the w-variables-$[L, \partial/\partial y] = 0$. We will view L as an operator on the sphere S. Let

$$\hat{\alpha} = \sum_{j=1}^{n} \alpha_j x_j/|x|, \quad then$$

$$\hat{\alpha} D = e^{-y}(\partial/\partial y - L);$$

and since $\hat{\alpha}^2 = -1$,

$$e^y D = \hat{\alpha}(\partial/\partial y - L) \tag{5}$$

Note that$\hat{\alpha}$is unitary and $L^* = L$. If we recall that

$$\Delta = e^{-2y}(\partial^2/\partial y^2 + (n-2)\partial/\partial y + \Delta_S), \tag{6}$$

where $\Delta_S$denotes the Laplace-Beltrami operator of the sphere. It follows from $D^* = D, \quad D^2 = -\Delta$ that

$$L(L + n - 2) = -\Delta_S \tag{7}$$

In general if $\psi \in C^\infty(R)$, then (6) implies that in polar coordinates $x = e^y w$,

$$e^{ts(y)} e^y D e^{-ts(y)} h = \hat{\alpha} A_t h \tag{8}$$

where $A_t = \partial/\partial y - (ts(y) + L)$.

Now we will prove the theorem.

Proof. Choose a small subset $A \subset U$ which will be decided later. From(3) and (4) we have

$$||e^{ts(y)} \nabla u||_{L^2(A)} + ||e^{ts(y)} u||_{L^{q_2}(A)}$$

$$\leq C||e^{ts(y)} \Delta u||_{L^p(R^n \setminus A)} + C||e^{ts(y)} \Delta u||_{L^p(A)}$$

On the other hand, from(1), the right hand side is bounded by

$$C||e^{ts(y)} \Delta u||_{L^p(R^n \setminus A)} + C||e^{ts(y)} (\nabla W) u||_{L^p(A)}$$

Integrating by parts, we find the second part of the above is bounded by

$$C||e^{ts(y)} W(\nabla u)||_{L^p(A)} + C||(\nabla e^{ts(y)}) W u||_{L^p(A)}$$

Hölder's inequality tells us the above is bounded by

$$C||e^{ts(y)}(\nabla u)||_{L^2(A)}||W||_{L^r(A)} + C||ts'(y) e^{ts(y)} u||_{L^2(A)}||W||_{L^r(A)}$$

If we sum all the terms we finally obtain

$$||e^{ts(y)} \nabla u||_{L^2(A)} + ||e^{ts(y)} u||_{L^{q_2}(A)}$$

$$\leq C||e^{ts(y)}\Delta u||_{L^p(R^n\setminus A)} + C||e^{ts(y)}(\nabla u)||_{L^2(A)}||W||_{L^r(A)}$$
$$+C||ts'(y)e^{ts(y)}u||_{L^2(A)}||W||_{L^r(A)}$$

The idea is to make cancellations of the last two terms on the right against the left hand side. If we choose A small as possible as $||W||_{L^r(A)} << 1/4$, and use the following $L^2 \to L^2$ estimate

$$t||e^{ts(y)}u||_{L^2(A)} \leq C||e^{ts(y)}\nabla u||_{L^2(A)} \tag{10}$$

After cancellations, we obtain the following:

$$||e^{ts(y)}\nabla u||_{L^2(A)} + ||e^{ts(y)}u||_{L^{q_2}(A)}$$
$$\leq C||e^{ts(y)}\Delta u||_{L^p(R^n\setminus A)}$$

Since $s(y)$ is radial and decreasing, choose a value $a \in \partial A$. Then

$$\begin{aligned} ||e^{ta}\nabla u||_{L^2(A)} + ||e^{ta}u||_{L^{q_2}(A)} &\leq ||e^{ts(y)}\nabla u||_{L^2(A)} + ||e^{ts(y)}u||_{L^{q_2}(A)} \\ &\leq C||e^{ta}\Delta u||_{L^p(R^n\setminus A)} \\ &\leq C' \end{aligned}$$

Letting t to infinity, we are forced to $u = 0$ identically in A.

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