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BETTI NUMBERS OVER ARTINIAN LOCAL RINGS

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I. Introduction

Every ring in this paper is assumed to be commutative and noetherian.

Let $(R, \underline{n})$ be a local ring with maximal ideal $\underline{n}$ and M be a finitely generated R-module. The ith Betti number $b_i^R(M)$ of M is the integer $\dim_{R/\underline{n}} \operatorname{Tor}_i^R(M, R/\underline{n})$, which is also the rank of the i^{th} module in the minimal R-free resolution of M. In this paper, we study the asymptotic behavior of the sequence of Betti numbers. We focus on the following two problems and develop tools for measuring nondecreasing and exponential growth of the Betti numbers.

PROBLEM 1.1. *(L. Avramov; [Av_1, 5.8]) Is the sequence $b_i^R(M)$ eventually nondecreasing for any finitely generated module M over the local ring R?*

PROBLEM 1.2. *(M. Ramras; [R_2]) Is it true that for an arbitrary finitely generated module over a local ring R, there are only two possibilities: either the sequence $b_i^R(M)$ is eventually constant, or $\lim_i b_i^R(M) = \infty$?*

Problem 1.1 is known to be true, if R is a Golod local ring [L_2, Corollary 6.5], $\underline{n}^3 = 0$ [L_1, Theorem B(3), Proposition3.9], or if R has a regular presentation with an integrally closed kernel [C_1,Theorem 1.1].

Besides the possible asymptotic behavior of Betti numbers as stated above we may introduce the growth of Betti numbers as follows:

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DEFINITION 1.3. [Av_3, Definition 1.1] Let M be a finitely generated module over the local ring $(R, \underline{n})$. Put

$$\beta_i^R(M) = \sum_{j=0}^{i} b_j^R(M).$$

The *complexity* of M, $\mathrm{cx}_R M$, is defined to be d, if $d-1$ is the smallest degree of a polynomial in i, which bounds $b_i^R(M)$ from above (the zero polynomial is of degree -1); if no such polynomial exists, then we set $\mathrm{cx}_R M = \infty$.

We say that the sequence of Betti numbers has *polynomial growth of degree* d, if there exist polynomials $p(x)$ and $q(x)$ with real coefficients, which are both degree d and have the same leading term such that we have the inequalities

$$p(i) \leq \beta_i^R(M) \leq q(i), \qquad i >> 0.$$

We say that the sequence of Betti numbers has *exponential growth*, if there exist real numbers $1 < A \leq B$ such that we have the inequalities

$$A^i \leq \beta_i^R(M) \leq B^i, \qquad i >> 0.$$

If we can replace $\beta_i^R(M)$ by $b_i^R(M)$ in the inequalities above, then we say the sequence of Betti numbers has *strong polynomial growth of degree* d, resp. *strong exponential growth.* An upper exponential bound B is known to exist for any finitely generated R-module (see, [L_1, Lemma in sec.1] or [Av_3, (2.5)]).

Over a complete intersection the sequence $b_i^R(M)$ has strong polynomial growth of degree one less than $cx_R M$ for a finitely generated module M [Av_2, Theorem 4.1]. The sequence of Betti numbers has strong exponential growth, if R is a Golod local ring and the sequence is strictly increasing [Av_3, Corollary 2.7], or if R has a regular presentation $(S, \underline{m})$ with an $\underline{m}$-primary $\underline{m}$-full kernel [C_2, Theorem 3.5].

In this paper we study exponential growth of Betti numbers over artinian local rings. By the Change of Tor Formula the results in the paper extend to the asymptotic behavior of Betti numbers over Cohen-Macaulay local rings. Using the length function of an artinian ring we calculate an upper bound for the number of generators of modules, this is then used to maximize the number of generators of sygyzy modules. Finally, applying a filtration of an ideal, which we call a Loewy series of an ideal, we derive an invariant $B(R)$ of an artinian local ring R, such that if $B(R) > 1$, then the sequence $b_i^R(M)$ of Betti numbers is strictly increasing and has strong exponential growth for any finitely generated non-free R-module M (Theorem 2.7).

II. Modules over artinian local rings

In this section we study the sequence of Betti numbers over artinian local rings and exhibit classes of artinian local rings, over which the sequence of Betti numbers is eventually nondecreasing and has strong exponential growth. First, we minimize the number of generators of a module when it is contained in IR^b (I is an ideal of an artinian local ring R). This is then used to maximize the lower exponential bound of the growth of Betti numbers. Our main points in this section are Lemma 2.4 and Proposition 2.6.

DEFINITION 2.1. Let $(R, \underline{n})$ be an artinian local ring with maximal ideal $\underline{n}$ and let I be an ideal of R. Let $l(I)$ denote the length of I and let $h(I)$ be the integer h satisfying $\underline{n}^h I \neq 0$ but $\underline{n}^{h+1} I = 0$. The Hilbert function $H_R(t)$ of R is the polynomial, $1 + e_1 t^1 + \cdots + e_{h(R)} t^{h(R)}$ where $e_i = \dim_{R/\underline{n}} \underline{n}^i / \underline{n}^{i+1}$. Let

$$\delta_k(R) := 2e_k + h(R) - l(R) - k, \qquad 1 \leq k \leq h(R),$$

and let $\Delta_1(R) := \delta_1(R)$ and for $2 \leq k \leq h(R)$

$$\Delta_k(R) := \frac{2e_k + e_{k-1} + \cdots + e_1 + h(R) - l(R) - k}{e_{k-1} + \cdots + e_1 + 1}.$$

We now quote results of Gasharov and Peeva.

LEMMA 2.2. [G-P, Lemma 2.1] *Let $(R,\underline{n})$ be an artinian local ring and let M be a finitely generated R-module such that $M \subseteq \underline{n}^n R^b$ $(b \geq 1,\ n \geq 0)$. Then*

$$\mu(M) \leq (l(\underline{n}^n) + n - h(R))b.$$

PROPOSITION 2.3. [G-P, Proposition 2.2] *Let $(R,\underline{n})$ be an artinian local ring and let M be a finitely generated R-module. Then*

$$\delta_1(R) b_i^R(M) \leq b_{i+1}^R(M), \qquad i \geq \mu(M).$$

We generalize Lemma 2.2 as follows.

LEMMA 2.4. *Let $(R,\underline{n})$ be an artinian local ring and I be an ideal of R. If M is a submodule of IR^b $(b \geq 1)$, then*

$$\mu(M) \leq (l(I) - h(I))b.$$

Proof. By induction on $h(I)$. If $h(I) = 0$, then both M and IR^b are vector spaces over $R/\underline{n}$. So $\mu(M) \leq l(I)b$. Now let $h(I) \neq 0$ and assume the induction hypothesis. Let $v_1, \cdots, v_s$ be a minimal generating set of M. We may assume that $v_1, \cdots, v_p$ $(p \leq s)$ are contained in a minimal generating set of IR^b and $v_{p+1}, \cdots, v_s$ are in $\underline{n}IR^b$.

Consider first the case when $p \leq (\mu(I)-1)b$. Let $M' = Rv_{p+1} + \cdots + Rv_s$, then $M' \subseteq \underline{n}IR^b$ such that $h(\underline{n}I) = h(I) - 1$. So by the induction hypothesis

$$\begin{aligned}\mu(M) &= p + \mu(M') \\ &\leq (\mu(I) - 1)b + (l(\underline{n}I) - h(\underline{n}I))b \\ &= (l(I) - h(I))b.\end{aligned}$$

Suppose now $p > (\mu(I)-1)b$. Set $q = \mu(I)b - p$, then $0 \leq q < b$. Let $v_1, \cdots, v_p, w_1, \cdots, w_q$ be a minimal generating set for IR^b. We may take w_is of the form

$$w_i = (0, \cdots, 0, a_i, 0, \cdots, 0), \qquad a_i \in I,$$

where a_i is in the k_ith place. Furthermore, permuting the summand of R^b, if necessary, we may assume that $k_i > b - q$ $(1 \leq i \leq q)$. That is, $w_i \in IR^q$ under the decomposition $R^b = R^{b-q} \oplus R^q$. Thus for $p + 1 \leq j \leq s$ we have

$$v_j = \sum_{i=1}^{p} r_{ij} v_i + \sum_{i=1}^{q} s_{ij} w_i, \qquad r_{ij}, s_{ij} \in \underline{n}.$$

Let $v'_j = \sum_{i=1}^{q} s_{ij} w_i$ and $M'' = Rv'_{p+1} + \cdots + Rv'_s$. Then $M = Rv_1 + \cdots + Rv_p + M''$ and $M'' \subseteq \underline{n}IR^q$. Applying the induction hypothesis for M'' we obtain

$$\begin{aligned} \mu(M) &= p + \mu(M'') \\ &\leq \mu(I)b - q + (l(\underline{n}I) - h(\underline{n}I)q \\ &= \mu(I)b - q + (l(\underline{n}I) - h(I) + 1)q \\ &\leq (l(I) - h(I))b. \end{aligned}$$

This finishes the proof of the lemma. □

DEFINITION 2.5. Let $(R, \underline{n})$ be a artinian local ring and I be an ideal of R. A sequence $\mathbf{I}$ of properly included ideals,

$$\mathbf{I}: \quad I = I_\alpha \subset I_{\alpha-1} \subset \cdots \subset I_1 \subset I_0 = R,$$

is called a *Loewy series* of I, if $\underline{n}I_i \subseteq I_{i+1}$ $(0 \leq i \leq \alpha - 1)$. We also define

$$\Delta(\mathbf{I}) := \frac{l(I) - 2l(\underline{n}I) + h(\underline{n}I)}{\mu(I_0) + \cdots + \mu(I_{\alpha-1})},$$

$$\beta(I) := \sup\{\, \Delta(\mathbf{I}) \mid \mathbf{I} \text{ is a Loewy series of } I \,\},$$

$$B(R) := \sup\{\, \beta(I) \mid I \text{ is an ideal of } R \,\}.$$

Note that for any ideal I of the artinian local ring $(R, \underline{n})$ a Loewy series of I exists. That is,

$$I = I_\alpha \subset (I : \underline{n}) \subset (I : \underline{n}^2) \subset \cdots \subset (I : \underline{n}^\alpha) = R$$

is a Loewy series of I. Note also $I_1 = \underline{n}$.

PROPOSITION 2.6. *Let $(R,\underline{n})$ be an artinian local ring with $B(R) > 1$ and let M be a finitely generated R-module. Then for any $1 < A < B(R)$ and $i \geq \mu(M)$*

$$Ab_i^R(M) \leq b_{i+1}^R(M).$$

Proof. We may assume M is not free, otherwise the assertion is trivial. There exists then an integer n, $1 \leq n \leq \mu(M)$, such that $b_n \geq b_{n-1}$. Since $B(R) > 1$, there exists an ideal I and a Loewy series, $\mathbf{I}: \ I = I_\alpha \subset I_{\alpha-1} \subset \cdots \subset I_1 \subset I_0 = R$, of I such that $\Delta(\mathbf{I}) > A$ for any $1 < A < B$.

Consider

$$IR^{b_n} \xrightarrow{d_n} \underline{n}IR^{b_{n-1}}.$$

Set $b = b_n$ and $d = d_n$ for convenience. Let I be minimally generated by $a_1, \cdots, a_e$ and $f_k = (0, \cdots, 0, a_i, 0, \cdots, 0) \in IR^b$ for $i = 1, \cdots, e$ and $k = 1, \cdots, eb$. Put $L = Rdf_1 + \cdots + Rdf_{eb}$. By Lemma 6.4 L can be generated by q elements, say, by $dw_1, \cdots, dw_q$ where $q \leq (l(\underline{n}I) - h(\underline{n}I))b_{n-1} \leq (l(\underline{n}I) - h(\underline{n}I))b$. Thus for $q+1 \leq j \leq eb$

$$df_j = \sum_{i=1}^{q} r_{ij} df_i, \qquad r_{ij} \in R.$$

Hence $u_j = f_j - \sum_{i=1}^{q} r_{ij} f_i$ is in $\ker d$.

Step 1. Let $V_0 = \{u_{q+1}, \cdots, u_{eb}\}$, and express V_0 as a disjoint union of two sets

$$V_0 = \{v_1^0, \cdots, v_{s_0}^0\} \dot\cup \{w_1^0, \cdots, w_{t_0}^0\}$$

where $v_i^0 \in \ker d - I_1 \ker d$ and $w_i^0 \in I_1 \ker d$. Note that $V_0 \subseteq \ker d$ is linearly independent modulo $\underline{n}IR^b$, and $eb - q = |V_0|$.

Step 2. Suppose that we have constructed $V_0, \cdots, V_k$ $(k \leq \alpha - 2)$ inductively, such that each V_i is contained in $I_i \ker d$ and is linearly independent modulo $\underline{n}IR^b$ $(0 \leq i \leq k)$ satisfying

$$eb - q \leq (\mu(I_0) + \cdots + \mu(I_{k-1}))b_{n+1} + |V_k|.$$

Express V_k as a disjoint union of two sets,

$$V_k = \{v_1^k, \cdots, v_{s_k}^k\} \dot\cup \{w_1^k, \cdots, w_{t_k}^k\}$$

where $v_i^k \in I_k \ker d - I_{k+1} \ker d$ and $w_i^k \in I_{k+1} \ker d$.

If $s_k \leq \mu(I_k)b_{n+1}$, then let $p_k = s_k$ and $V_{k+1} = \{w_1^k, \cdots, w_{t_k}^k\}$. Otherwise take a maximal subset, say, $v_1^k, \cdots v_{p_k}^k$ of $v_1^k, \cdots, v_{s_k}^k$ ($p_k < s_k$), which are linearly independent modulo $\underline{n}I_k \ker d$. Join $x_1^k, \cdots, x_{q_k}^k$, such that $p_k + q_k = \mu(I_k \ker d) \leq \mu(I_k)b_{n+1}$ and $v_1^k, \cdots, v_{p_k}^k, x_1^k, \cdots, x_{q_k}^k$ are a basis of $I_k \ker d$. Then for $p_k + 1 \leq j \leq s_k$

$$v_j^k = \sum_{i=1}^{p_k} r_{ij} v_i^k + \sum_{i=1}^{q_k} s_{ij} x_i^k, \qquad r_{ij}, s_{ij} \in R.$$

We claim that $s_{ij} \in \underline{n}$ for all i and j. Suppose not, say, s_{1p_k+1} is not in $\underline{n}$. Then

$$x_1^k = 1/s_{1p_k+1}(v_{p_k+1}^k - \sum_{i=1}^{p_k} r_{ip_k+1} v_i^k - \sum_{i=2}^{q_k} s_{ip_k+1} x_i^k).$$

Substituting $v_{p_k+1}^k$ for x_1^k, we see that $v_1^k, \cdots v_{p_k+1}^k, x_2^k, \cdots, x_{q_k}^k$ are a basis of $I_k \ker d$. This contradicts the choice of $v_1^k, \cdots, v_{p_k}^k$.

Now consider

$$z_j^k = v_j^k - \sum_{i=1}^{p_k} r_{ij} v_i^k = \sum_{i=1}^{q_k} s_{ij} x_i^k$$

contained in $\underline{n}I_k \ker d$ so contained in $I_{k+1} \ker d$ and let

$$V_{k+1} = \{w_1^k, \cdots, w_{t_k}^k, z_{p_k+1}^k, \cdots, z_{s_k}^k\}$$

. Then V_{k+1} is contained in $I_{k+1} \ker d$ and is linearly independent modulo $\underline{n}IR^b$ satisfying

$$\begin{aligned} eb - q &\leq (\mu(I_0) + \cdots + \mu(I_{k-1}))b_{n+1} + s_k + t_k \\ &= (\mu(I_0) + \cdots + \mu(I_{k-1}))b_{n+1} + p_k + (s_k - p_k) + t_k \\ &\leq (\mu(I_0) + \cdots + \mu(I_k))b_{n+1} + |V_{k+1}|. \end{aligned}$$

Step 3. Inductively, we have constructed $V_1, \cdots, V_{\alpha-1}$ such that

$$eb - q \leq (\mu(I_0) + \cdots + \mu(I_{\alpha-2}))b_{n+1} + |V_{\alpha-1}|.$$

Note that $V_{\alpha-1}$ is contained in $I_{\alpha-1} \ker d$ and is linearly independent modulo $\underline{n}IR^b$. So it is linearly independent modulo $\underline{n}I_{\alpha-1} \ker d$, since $\underline{n}I_{\alpha-1} \ker d \subseteq I \ker d \subseteq \underline{n}IR^b$. Hence

$$|V_{\alpha-1}| \leq \mu(I_{\alpha-1} \ker d) \leq \mu(I_{\alpha-1})b_{n+1}.$$

Therefore

$$\begin{aligned}(\mu(I_0) + \cdots + \mu(I_{\alpha-1}))b_{n+1} &\geq eb_n - q \\ &= \mu(I)b_n - q \\ &\geq \mu(I)b_n - (l(\underline{n}I) - h(\underline{n}I))b_n \\ &= (l(I) - 2l(\underline{n}I) + h(\underline{n}I))b_n.\end{aligned}$$

In particular, $b_{n+1} \geq b_n$ and we can iterate the above argument. □

In terms of exponential growth and nondecreasing of Betti numbers, Proposition 2.6 can be restated as follows.

PROPOSITION 2.7. *Let $(R, \underline{n})$ be an artinian local ring and M be a finitely generated R-module.*

(1) *If $B(R) > 1$, then for any non-free R-module M the sequence $b_i^R(M)$, $i \geq \mu(M)$ of Betti numbers is strictly increasing and has strong exponential growth with a lower exponential bound A for any $1 < A < B(R)$.*
(2) *If there exist an ideal I of R and a Loewy series, $\mathbf{I} : I = I_\alpha \subset I_{\alpha-1} \subset \cdots \subset I_1 \subset I_0 = R$, of I such that $\Delta(\mathbf{I}) \geq 1$, then*

$$\Delta(\mathbf{I})b_i^R(M) \leq b_{i+1}^R(M).$$

So the sequence $b_i^R(M)_{i \geq \mu(M)}$ of Betti numbers is nondecreasing for any finitely generated R-module M.

Proof. Immediate from Proposition 2.6. □

REMARK 2.8. If $\Delta(\mathbf{I}) \geq 1$ for a Loewy series, $\mathbf{I} : I = I_\alpha \subset I_{\alpha-1} \subset \cdots \subset I_1 \subset I_0 = R$, of an ideal I in an artinian local ring $(R, \underline{n})$. Then the denominator of $\Delta(\mathbf{I})$ satisfies the inequality,

$$\begin{aligned}\mu(I_0) + \cdots + \mu(I_{\alpha-1}) &\geq l(R) - l(\underline{n}I_{\alpha-1}) \\ &\geq l(R) - l(I).\end{aligned}$$

Hence for $\Delta(\mathbf{I}) \geq 1$, it is necessary that

$$\frac{l(I) - 2l(\underline{n}I) + h(\underline{n}I)}{l(R) - l(I)} \geq 1,$$

equivalently,

$$2\mu(I) + h(\underline{n}I) - l(R) \geq 0.$$

If we interpret Proposition 2.6 and 2.7 in terms of the Hilbert's coefficients e_i, then we obtain the following theorem which also extend Proposition 2.3.

THEOEM 2.9. *Let $(R, \underline{n})$ be an artinian local ring with $\Delta_k(R) \geq 1$ for some k and let M be a finitely generated R-module. Then*

$$\Delta_k(R) b_i^R(M) \leq b_{i+1}^R(M).$$

So if $\Delta_k(R) > 1$, then for any non-free module M the sequence $b_i^R(M)$, $i \geq \mu(M)$ of Betti numbers is strictly increasing and has strong exponential growth with a lower exponential bound $A = \Delta_k(R)$. And if $\Delta_k(R) \geq 1$, equivalently, $\delta_k(R) \geq 1$, then for any finitely generated R-module M the sequence $b_i^R(M)_{i \geq \mu(M)}$ is nondecreasing.

Proof. For $I = \underline{n}^k$ we have an obvious Loewy series,

$$\mathbf{I} : \underline{n}^k \subset \underline{n}^{k-1} \subset \cdots \underline{n} \subset R,$$

of $\underline{n}^k$. In this case $\Delta(\mathbf{I}) = \Delta_k$ and the assertion of the theorem follows from Proposition 2.6 and 2.7. □

REMARK 2.10. The result of Gasharov and Peeva (Proposition 2.3) has been applied for artinian local rings of 'lower' lengths (cf. [G-P, Theorem 1.1]). On the other hand the generalized form (Theorem 2.9) can be used for artinian local rings of 'higher' embedding dimensions, in which case e_i may increase fast. For example, let $S = k[[x, y, z]]$, the power series ring in three variables x, y and z over the field k and let $R = S/I$ where I is the homogeneous ideal generated by all monomials of degree 3 except xyz, then the Hilbert's polynomial is $H_R(t) = 1 + 3t + 6t^2 + t^3$. Hence $\Delta_1(R) = -3$ and $\Delta_2(R) = 5/4$. So the sequence $b_i^R(M)$ of any finitely generated non-free R-module M is strictly increasing and has strong exponential growth with a lower exponential bound $5/4$.

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