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GENERIC MINIMAL SUBMANIFOLDS WITH PARALLEL SECTION IN THE NORMAL BUNDLE IMMERSED IN A COMPLEX PROJECTIVE SPACE

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1. Introduction

Let CP^m denote the complex projective space of complex dimension m (real dimension $2m$) equipped with the standard symmetric metric g normalized so that the maximum sectional curvature is four. We denote by J the almost complex structure of CP^m. Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m. We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M. If $JT_x(M)^\perp \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of CP^m. Any real hypersurface of CP^m is obviously a generic submanifold of CP^m.

In [2] we proved that if the minimum of the sectional curvature of a compact real minimal hypersurface of CP^m is $1/(2m-1)$, then M is the geodesic hypersphere. This result was generalized in [8] to the case of M is a generic submanifold with flat normal connection.

The purpose of the present paper is to prove a following generalization of theorems in [2] and [8].

THEOREM. *Let M be a compact n-dimensional generic minimal submanifold of CP^m with nonvanishing parallel section in the normal bundle. If the minimum of the sectional curvatures of M is $1/n$, the $2m = n+1$ and M is the geodesic minimal hypersphere.*

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2. Preliminaries

Let M be a real n-dimensional submanifold of CP^m. We denote by the same g the Riemannian metric tensor field induced on M from that of CP^m. The operator of covariant differentiation with respect to the Levi-Civita connection in CP^m (resp. M) will be denoted by $\tilde{\nabla}$ (resp. ∇). Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^\perp$ of M. A and B appearing here are both called the second fundamental forms of M and are related by $g(B(X,Y),V) = g(A_V X, Y)$.

The covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of the normal vector* V. If the second fundamental form is parallel in any direction, it is said to be *parallel*. If $\mathrm{Tr} A_V = 0$ for any vector field V normal to M, then M is said to be *minimal*, where Tr denotes the trace of the operator. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M.

In the sequel, we assume that M is a generic submanifold of CP^m. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x(M) + JT_x(M)^\perp$ at each point x of M , where $H_x(M)$ denotes the orthogonal complement of $JT_x(M)^\perp$ in $T_x(M)$. Then we see that $H_x(M)$ is a holomorphic subspace of $T_x(M)$.

For any vector field X tangent to M, we put

$$JX = PX + FX, \tag{2.1}$$

where PX is the tangential part and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. We have $P^2 = -I - JF$ and $FP = 0$. We define the covariant derivatives of P and F by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$ and $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$ respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + JB(X,Y), \quad (\nabla_X F)Y = -B(X,PY), \tag{2.2}$$

$$\nabla_X JV = -PA_V X + JD_X V, \qquad B(X,JY) = -FA_V X. \tag{2.3}$$

For any vector fields X and Y in $JT(M)^\perp$ we obtain

$$A_{FX}Y = A_{FY}X. \tag{2.4}$$

Hence we have

$$A_V JU = A_U JV \tag{2.5}$$

for any vector fields U and V normal to M.

We notice that P satisfies $P^3 + P = 0$, and hence P defines an f-structure on M (see [7]).

We denote by R and S the Riemannian curvature tensor and the Ricci tensor of M respectively. Then

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ &+ 2g(X,PY)PZ + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \end{aligned} \tag{2.6}$$

$$\begin{aligned} S(X,Y) =& (n-1)g(X,Y) + 3g(PX,PY) \\ &+ \sum \mathrm{Tr} A_a g(A_a X,Y) - \sum g({A_a}^2 X,Y), \end{aligned} \tag{2.7}$$

where A_a is the second fundamental form in the direction of v_a, $\{v_a\}$ being an orthonormal frame for $T_x(M)^\perp$. The Codazzi equation of M is given by

$$\begin{aligned} g((\nabla_X A)_V Y, Z) &- g((\nabla_Y A)_V X, Z) \\ &= g(PY,Z)g(FX,V) - g(PX,Z)g(FY,V) \\ &+ 2g(X,PY)g(FZ,V). \end{aligned} \tag{2.8}$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have equation of the Ricci

$$\begin{aligned} &g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X,Y) \\ &\qquad = g(FY,V)g(FX,U) - g(FX,V)g(FY,U). \end{aligned} \tag{2.9}$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be *flat*. We can see that the normal connection of M is flat if and only if there exist locally $2m-n$ mutually orthogonal unit normal vector fields v_a such that each of the v_a is parallel (cf. [1]).

3. Integral formulas

Let M be an n-dimensional generic minimal submanifold of CP^m. For any vector field X tangent to M we have (see [6])

$$\begin{aligned} &\mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div}X)X) \\ &= S(X,X) + \frac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (\mathrm{div}X)^2, \end{aligned} \tag{3.1}$$

where $L(X)g$ denotes the Lie derivative of g with respect to the vector field X, and $|Y|$ denotes the length of a tensor field Y on M with respect to g.

Suppose now that U is a parallel section in the normal bundle of M, that is, $DU = 0$. Then (2.3) implies that $\nabla_X JU = -PA_U X$. We denote by $\{e_i\}$ an orthonormal frame on M. Then we have $\mathrm{div}JU = \sum g(\nabla_i JU, e_i) = -\mathrm{Tr}PA_U = 0$, since P is skew-symmetric and A_U is symmetric, where ∇_i denotes the covariant differentiation in the direction of e_i. Thus (3.1) implies

$$\mathrm{div}(\nabla_{JU} JU) = S(JU,JU) + \frac{1}{2}|L(JU)g|^2 - |\nabla JU|^2. \tag{3.2}$$

On the other hand, we have, from (2.5)

$$|\nabla JU|^2 = \mathrm{Tr} A_U^2 - \sum g({A_a}^2 JU, JU).$$

Substituting this equation and (2.7) into (3.2), we have

$$\text{(3.3)} \qquad \mathrm{div}(\nabla_{JU} JU) = (n-1)g(JU, JU) - \mathrm{Tr} A_U^2 + \frac{1}{2}|L(JU)g|^2.$$

Since we have

$$(L(JU)g)(X,Y) = g(\nabla_X JU, Y) + g(\nabla_Y JU, X) = g((A_U P - P A_U)X, Y),$$

we obtain

$$\frac{1}{2}|L(JU)g|^2 = |[P, A_U]|^2 = 2\{\mathrm{Tr}(A_U P)^2 - \mathrm{Tr} A_U^2 P^2\}.$$

In the following, we assume that there exists a nonvanishing parallel section U in the normal bundle of M.

To simplify the notation, we put $A = A_U$, $|U| = 1$. Then $\mathrm{Tr} A = 0$. From the equation of Codazzi (2.8), we see

$$\sum (\nabla_i A) e_i = 0.$$

LEMMA 3.1. *The restricted Laplacian of A satisfies*

$$\begin{aligned} g(\nabla^2 A, A) = &\sum g((R(e_i, e_j)A)e_i, Ae_j) \\ &+ (3/2)|[P, A]|^2 - 3\mathrm{Tr} A^2 + 3(p-1), \end{aligned}$$

where $p = \mathrm{codim} M = 2m - n$.

Proof. From (2.8) and (3.4) we have

$$\text{(3.5)} \qquad \begin{aligned} g(\nabla^2 A, A) &= \sum g((\nabla_i \nabla_i A)e_j, Ae_j) \\ &= \sum g((R(e_i, e_j)A)e_i, Ae_j) + 3\mathrm{Tr}(AP)^2 \\ &\quad - 3\sum g(A_a J v_a, AJU) \end{aligned}$$

Since U is parallel, (2.9) implies

$$\sum\{g(AJv_a, AJv_a) - g(A_aJv_a, AJU)\} = p - 1.$$

Therefore, we have

$$3\mathrm{Tr}(AP)^2 - 3\sum g(A_aJv_a, AJU) = (3/2)\|[P,A]\|^2 - 3\mathrm{Tr}A^2 + 3(p-1).$$

Substituting this equation into (3.5), we have our assertion.

From (3.3) and Lemma 3.1 we have

LEMMA 3.2. *We have the following equation:*

$$-g(\nabla^2 A, A) - 2(n-p) + \frac{1}{2}\|[P,A]\|^2 + p - 1$$
$$= -\sum g((R(e_i, e_j)A)e_i, Ae_j) + \mathrm{Tr}A^2 - 2\mathrm{div}(\nabla_{JU}JU).$$

LEMMA 3.3. *We have the inequality* $|\nabla A|^2 \geq 2(n-p)$. *The equality holds if and only if* $(\nabla_X A)Y = g(PX,Y)JU + g(Y,JU)PX$.

Proof. Put $T(X,Y) = (\nabla_X A)Y - g(PX,Y)JU - g(Y,JU)PX$. We compute the square of the length of T and obtain $|T|^2 = |\nabla A|^2 - 2(n-p) \geq 0$ by (2.8). Thus we have our assertion.

LEMMA 3.4. *Let M be a compact n-dimensional generic minimal submanifold of CP^m. If M admits a nonvanishing parallel section in the normal bundle, then*

$$0 \leq \int_M \{|\nabla A|^2 - 2(n-p) + \frac{1}{2}\|[P,A]\|^2 + (p-1)\}{}^*1$$
$$= \int_M \{\mathrm{Tr}A^2 - \sum g((R(e_i,e_j)A)e_i, Ae_j)\}{}^*1.$$

Proof. We have

$$\frac{1}{2}\Delta \mathrm{Tr}A^2 = g(\nabla^2 A, A) + |\nabla A|^2.$$

Thus we have

$$-\int_M g(\nabla^2 A, A){}^*1 = \int_M |\nabla A|^{2*}1.$$

Therefore, Lemmas 3.1 and 3.2 prove our assertion.

4. Proof of theorem

First of all, we prove

LEMMA 4.1. *Let M be a compact n-dimensional generic minimal submanifold of CP^m with nonvanishing parallel section in the normal bundle. If the minimum of the sectional curvatures of M is $1/n$, then $p = 1$, $|\nabla A|^2 = 2(n-1)$ and $PA = AP$.*

Proof. We choose an orthonormal frame $\{e_i\}$ of M such that $Ae_i = \lambda_i e_i$ $(i = 1, \cdots, n)$. We denote by K_{ij} the sectional curvature of M spanned by e_i and e_j. Then we have

$$\begin{aligned}
&\sum g((R(e_i,e_j)A)e_i, Ae_j) \\
&\quad = \sum \{g(R(e_i,e_j)Ae_i, Ae_j) - g(AR(e_i,e_j)e_i, Ae_j)\} \\
&\quad = \frac{1}{2}\sum(\lambda_i - \lambda_j)^2 K_{ij} \\
&\quad \geq (1/2n)\sum(\lambda_i - \lambda_j)^2 = \mathrm{Tr}A^2.
\end{aligned}$$

Consequently, we see

$$\mathrm{Tr}A^2 - \sum g((R(e_i,e_j)A)e_i, Ae_j) \leq 0.$$

From this and Lemma 3.4 we have our assertion.

Model space. We consider the standard fibration
$$S^1 \longrightarrow S^{2n+1} \longrightarrow CP^n,$$
where S^k denotes the k-dimensional Euclidean sphere of curvature 1. In S^{2n+1} we have the family of generalized clifford surfaces whose spheres lie in complex subspaces (cf. [3]) :
$$M_{2p+1,2q+1} = S^{2p+1}(((2p+1)/2n)^{\frac{1}{2}}) \times S^{2q+1}(((2q+1)/2n)^{\frac{1}{2}}),$$
where $p + q = n - 1$. Then we have a fibration
$$S^1 \longrightarrow M_{2p+1,2q+1} \longrightarrow M^C_{p,q},$$
compatible the standard fibration. In the special case $p = 0$, $M^C_{0,n-1}$ is called a *geodesic minimal hypersphere* (see [5]), and is a homogeneous, positively curved manifold diffeomorphic to the sphere (see [3], [5]).

The minimum of the sectional curvature of $M^C_{0,n-1}$ is $1/n$, and that of $M^C_{p,q}(p,q \geq 1)$ is zero.

Proof of Theorem. From Lemma 4.1 we see that M is a real hypersurface of CP^m. We also have $PA = AP$. Thus, from a theorem of [4] we see that M is $M^C_{p,q}$. Since the minimum of the sectional curvature of M is $1/n$, we see that M is the geodesic minimal hypersphere. Consequently, we have our assertion.

COROLLARY 4.1([8]). *Let M be a compact n-dimensional generic minimal submanifold of CP^m with flat normal connection. If the minimum of the sectional curvatures of M is $1/n$, then $2m = n + 1$ and M is the geodesic minimal hypersphere $M^C_{0,(n-1)/2}$.*

COROLLARY 4.2([2]). *Let M be a compact real minimal hypersurface. If the minimum of the sectional curvatures of M is $1/(2m-1)$, then M is the geodesic minimal hypersphere $M^C_{0,m-1}$.*

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