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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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1. Introduction

In the recent several years the theory of impulsive differential equations has made a rapid progress (see [1] and [2] and the references there). The questions of stability and periodicity of the solutions of these equations have been elaborated in sufficient details while their asymptotic behaviour has been little studied. In the present paper the asymptotic behaviour of the solutions of linear impulsive differential equations is investigated, generalizing the results of J. W. Macki and J. S. Muldowney, 1970 [3], related to ordinary differential equations without impulses.

2. Preliminary notes

Consider the linear impulsive differential equation

$$\begin{aligned} x' &= A(t)x\ , \quad t \neq \tau_k, \\ x^+ &= A_k x\ , \quad t = \tau_k, \end{aligned} \tag{1}$$

where $t \in [0, w) \subset \mathbb{R}$; $x = \mathrm{col}(x_1, \cdots, x_n)$ belongs to the n-dimensional complex vector space $\mathbb{C}^n$; $A(t)$, $A_k \in \mathbb{C}^{n\times n}$ for $k \in \mathbb{N} = \{1, 2, \cdots\}$ and $t \in [0, w)$; $\mathbb{C}^{n\times m}$ is the space of $n \times m$ complex-valued matrices.

Assume the following conditions (C) satisfied:

C1. $\{\tau_k\} \subset (0, w)$, $\tau_{k-1} < \tau_k$ $(k \in \mathbb{N})$, $\lim\limits_{k\to\infty} \tau_k = w$.

C2. The function $A : [0, w) \to \mathbb{C}^{n\times n}$ is locally integrable in $[0, w)$.

C3. $\det A_k \neq 0$ $\quad (k \in \mathbb{N})$.

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Denote by $APC([0, w), \mathbb{C}^{n\times m})$ the class of functions $\xi : [0, w) \to \mathbb{C}^{n\times m}$ which are absolutely continuous in the intervals $[0, \tau_1], (\tau_k, \tau_{k+1}]$ $(k \in \mathbb{N})$, for $t = \tau_k$ have discontinuities of the first kind and are continuous from the left.

The function $x(t)$ is said to be a *solution* of (1) if $x \in APC([0, w), \mathbb{C}^n)$, for $t \in [0, w)$, $t \neq \tau_k$ it satisfies the equation $x' = A(t)x$ almost everywhere ($a.e.$) and for $t = \tau_k$ — the condition $x(\tau_k^+) = A_k x(\tau_k)$.

In the present paper we shall find sufficient conditions under which the matrices $A(t), A_k$ $(k \in \mathbb{N})$ belong to one of the following classes:

Ω_0–the class of matrices $A(t), A_k$ $(k \in \mathbb{N})$ such that equation (1) has a nontrivial solution $x_0(t)$ satisfying $\lim_{t\to w} |x_0(t)| = 0$;

Ω_∞–the class of matrices $A(t), A_k$ $(k \in \mathbb{N})$ such that equation (1) has a solution $x_\infty(t)$ satisfying $\lim_{t\to w} |x_\infty(t)| = +\infty$;

ΩP_0–the class of matrices $A(t), A_k$ $(k \in \mathbb{N})$ such that equation (1) has a nontrivial solution $x_0(t)$ satisfying $\lim_{t\to w} |Px_0(t)| = 0$, where P is a projector in $\mathbb{C}^n$.

Here $|\cdot|$ is some fixed norm in $\mathbb{C}^n$ (which, of course, is topologically equivalent to the Euclidean norm $\|\cdot\|$).

3. Main results

LEMMA 1. *Let conditions* (C) *hold, the sequence* $\{t_k\} \subset [0, w)$ *and* $\lim_{k\to\infty} t_k = w$.

(a) *Suppose that* $0 \leq \lim_{k\to\infty} \|x(t_k)\| < \infty$ *exists for any solution* $x(t)$ *of (1). Then there exists a nontrivial solution* $x_0(t)$ *of (1) such that*

$$\lim_{k\to\infty} \|x_0(t_k)\| = 0$$

iff

$$\lim_{k\to\infty} M(t_k) = -\infty,$$

where

$$M(t) = \int_0^t Re\ Tr\ A(\tau)d\tau + \sum_{0<\tau_k<t} \log|\det A_k|.$$

(b) Suppose that $0 \leq \lim_{k\to\infty} \|x(t_i)\| \leq \infty$ *exists for any solution* $x(t)$ *of (1). Then there exists a solution* $x_\infty(t)$ *of (1) such that*

$$\lim_{k\to\infty} \|x_\infty(t_k)\| = +\infty$$

if

$$\lim_{k\to\infty} M(t_k) = +\infty. \qquad (1)$$

Proof. (a) Since

$$0 \leq \lim_{k\to\infty} x^*(t_k)x(t_k) < +\infty$$

exists for any solution $x(t)$ of (1), it is easily seen that

$$0 \leq |\lim_{k\to\infty} x^*(t_k)y(t_k)| < \infty$$

exists for any two solutions $x(t), y(t)$ of (1). Thus if $X(t)$ is a fundamental matrix of (1), then $\lim_{k\to\infty} X^*(t_k)X(t_k) = H$ exists and the matrix $H \in \mathbb{C}^{n\times n}$ is Hermitian. Also,

$$\lim_{k\to\infty} |\det X(t_k)|^2 = \lim_{k\to\infty} \det X^*(t_k)X(t_k) = \det H.$$

Hence, by the formula of Liouville-Jacobi for impulsive linear equations [1], p.46

$$\det X(t) = \det X(0) \prod_{0<\tau_k<t} \det A_k \exp\left(\int_0^t A(\tau)d\tau\right)$$

whence

$$|\det X(t_k)| = |\det X(0)|e^{M(t_k)}.$$

Therefore,

$$-\infty \leq \lim_{k\to\infty} M(t_k) < +\infty$$

exists and

$$\det H = \lim_{k\to\infty} |\det X(0)|^2 e^{2M(t_k)}.$$

Since $x(t) = X(t)\xi$ is a solution of (1) for any $\xi \in \mathbb{C}^n$, then

$$0 \leq \lim_{k\to\infty} \|x(t_k)\|^2 = \lim_{k\to\infty} \xi^* X^*(t_k) X(t_k)\xi = \xi^* H\xi,$$

i.e., H is nonnegatively definite. That is why,

$$\exists \xi \in \mathbb{C}^n,\ \xi \neq 0 \text{ such that } \xi^* H\xi = 0 \Leftrightarrow \det H = 0,$$

which is satisfied iff

$$\lim_{k\to\infty} M(t_k) = -\infty.$$

(b) Let (2) hold and suppose that $\lim_{k\to\infty} \|x(t_k)\| < +\infty$ any solution $x(t)$ of (1). Then, in view of what was proved in part (a), $\lim_{k\to\infty} M(t_k) < \infty$, which contradicts (2). ■

THEOREM 1. *Let conditions (C) hold.*

(a) Suppose that equation (1) is stable and each solution $x(t)$ enjoys the property:

$$\text{If } \liminf_{t\to w} |x(t)| = 0, \text{ then } \lim_{t\to w} |x(t)| = 0. \tag{3}$$

Then

$$(A(t), A_k) \in \Omega_0 \Leftrightarrow \liminf_{t\to w} M(t) = -\infty.$$

(b) Suppose that the solutions $x(t)$ of (1) enjoy the property:

$$\text{If } \limsup_{t\to w} |x(t)| = +\infty, \text{ then } \lim_{t\to w} |x(t)| = +\infty. \tag{4}$$

Then

$$(A(t), A_k) \in \Omega_\infty \Rightarrow \limsup_{t\to w} M(t) = +\infty.$$

Proof. (a) Suppose that there is no nontrivial solution $x(t)$ of (1) for which $\lim_{t\to w} |x(t)| = 0$. Then by (3) $\liminf_{t\to w} |x(t)| > 0$ for any nontrivial solution and also $\liminf_{t\to w} \|x(t)\| = 0$ for any such solution. By Lemma 1 we cannot have

$$\lim_{k\to\infty} M(t_k) = -\infty$$

for any sequence $\{t_k\}$ convergent to w. That is why $\liminf_{t\to w} M(t) > -\infty$.

Conversely, suppose that $\lim_{t\to w} |x_0(t)| = 0$ for some nontrivial solution $x_0(t)$ of (1). Let $X(t)$ be a fundamental matrix of (1) in which $x_0(t)$ is one of the columns. Since all solutions of (1) are bounded, then $\lim_{t\to w} |\det X(t)| = 0$. Then from the equality

$$|\det X(t)| = |\det X(0)| \exp M(t)$$

it follows that

$$\lim_{t\to w} M(t) = -\infty.$$

(b) Suppose that $\limsup_{t\to\infty} M(t) = +\infty$. If we suppose that all solutions of (1) are bounded, then we must have

$$\limsup_{t\to w} M(t) < +\infty$$

by part (b) of Lemma 1 and thus we get to a contradiction. Hence there exists a solution $x_\infty(t)$ of (1) such that $\limsup_{t\to w} |x_\infty(t)| = +\infty$ which by (4) implies that

$$\lim_{t\to w} |x_\infty(t)| = +\infty \qquad \blacksquare$$

COROLLARY 1.1. *Let conditions (C) hold. If equation (1) is stable and satisfies condition (3), then*

$$\liminf_{t\to w} M(t) = -\infty \;\Rightarrow\; \lim_{t\to w} M(t) = -\infty.$$

Proof. $\liminf_{t\to w} M(t) = -\infty$ implies the existence of a nontrivial solution $x_0(t)$ for which $\lim_{t\to w} |x_0(t)| = 0$. Then according to what was proved in part (a) of Theorem 1 we shall have

$$\lim_{t\to w} M(t) = -\infty. \qquad \blacksquare$$

COROLLARY 1.2. *Let conditions (C) hold.*

(a) Suppose that $0 \le \lim_{t\to w} |x(t)| < +\infty$ *exists for any solution* $x(t)$ *of equation (1). Then*

$$(A(t), A_k) \in \Omega_0 \;\Leftrightarrow\; \lim_{t\to w} M(t) = -\infty.$$

(b) Suppose that $0 \le \lim_{t\to w} |x(t)| \le +\infty$ *exists for any solution* $x(t)$ *of equation (1). Then*

$$(A(t), A_k) \in \Omega_\infty \;\Leftrightarrow\; \lim_{t\to w} M(t) = +\infty.$$

COROLLARY 1.3. *Suppose that conditions (C) hold and that equation (1) is uniformly stable. Then*

$$(A(t), A_k) \in \Omega_0 \;\Leftrightarrow\; \lim_{t\to w} M(t) = -\infty.$$

Proof. It suffices to show that if (1) is uniformly stable, then condition (3) is met. The uniform stability means that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x(t_0)| < \delta(\varepsilon)$, then $|x(t)| < \varepsilon$ for all $t \in [t_0, w)$. If $\liminf_{t\to w} |x(t)| = 0$, then for each $\varepsilon > 0$ there exists $t_\varepsilon \in (0, w)$ such that $|x(t_\varepsilon)| < \delta(\varepsilon)$, hence $|x(t)| < \varepsilon$ for $t \in [t_\varepsilon, w)$. Thus $\lim_{t\to w} |x(t)| = 0$. ■

REMARK 1. Corollary 1.1 implies, in particular, that if

$$-\infty = \liminf_{t\to w} M(t) < \limsup_{t\to w} M(t)$$

then either equation (1) is unstable, or it is stable and does not enjoy property (3). Thus equation (1) is neither asymptotically nor uniformly stable.

Suppose that the matrix $\Gamma(t) \in APC([0,w), \mathbb{C}^{n\times n})$ is nonsingular for $t \in [0, w)$ and $\det \Gamma(\tau_k^+) \neq 0 \quad (k \in \mathbb{N})$. Let $x(t)$ be a solution of equation (1). Then the function $y(t) = \Gamma(t)x(t)$ is a solution of the equation

$$\begin{aligned} y' &= B(t)y \quad , \quad t \neq \tau_k \ , \\ y^+ &= B_k y, \quad t = \tau_k \ , \end{aligned} \tag{5}$$

where

$$\begin{aligned} B(t) =& \Gamma'(t)\Gamma^{-1}(t) + \Gamma(t)A(t)\Gamma^{-1}(t), \\ B_k =& \Gamma(\tau_k^+)A_k\Gamma^{-1}(\tau_k). \end{aligned} \tag{6}$$

This fact immediately implies

THEOREM 2. *Let the matrix $\Gamma(t) \in APC([0,w),\mathbb{C}^{n\times n})$ be nonsingular for $t \in [0,w)$ and* $\det\Gamma(\tau_k^+) \neq 0 \quad (k \in \mathbb{N})$.

(a) If $|\Gamma(t)x| \leq K|x|$ for some constant $K > 0$ and all $x \in \mathbb{C}^n$ and $t \in [0,w)$, then

$$\begin{aligned} (A(t), A_k) \in \Omega_0 \ &\Rightarrow \ (B(t), B_k) \in \Omega_0, \\ (A(t), A_k) \in \Omega_\infty \ &\Leftarrow \ (B(t), B_k) \in \Omega_\infty. \end{aligned}$$

(b) If $|\Gamma(t)x| \geq k|x|$ for some constant $k > 0$ and all $x \in \mathbb{C}^n$ and $t \in [0,w)$, then

$$\begin{aligned} (B(t), B_k) \in \Omega_0 \ &\Rightarrow \ (A(t), A_k) \in \Omega_0, \\ (B(t), B_k) \in \Omega_\infty \ &\Leftarrow \ (A(t), A_k) \in \Omega_\infty. \end{aligned}$$

(c) If $k|x| \leq |\Gamma(t)x| \leq K|x|$ for some constants $0 < k \leq K$ and all $x \in \mathbb{C}^n$ and $t \in [0,w)$, then

$$\begin{aligned} (B(t), B_k) \in \Omega_0 \ &\Leftrightarrow \ (A(t), A_k) \in \Omega_0, \\ (B(t), B_k) \in \Omega_\infty \ &\Leftrightarrow \ (A(t), A_k) \in \Omega_\infty. \end{aligned}$$

(d) If $|\Gamma(t)x| \geq k|Px|$ for some constant $k > 0$ and all $x \in \mathbb{C}^n$ and $t \in [0,w)$, where P is a projector, then

$$(B(t), B_k) \in \Omega_0 \Rightarrow (A(t), A_k) \in \Omega P_0.$$

Now together with equation (1) consider the corresponding adjoint equation

$$y' = -A^*(t)y\ ,\ t \neq \tau_k,$$
$$y^+ = A_k^{*-1}y,\ t = \tau_k. \tag{7}$$

We shall note ([1], p.84) that if $x(t)$ and $y(t)$ are solutions respectively of (1) and (7), then

$$y^*(t)x(t) \equiv \ const \ \equiv y^*(0)x(0) \qquad (t \in [0, w)).$$

Thus, if $x_0(t)$ is a nontrivial solution of (1) such that

$$\lim_{t \to w} |x_0(t)| = 0$$

then there exists a solution $y_\infty(t)$ of (7) for which

$$1 \equiv y_\infty^*(t)x_0(t) \leq \|y_\infty(t)\| \|x_0(t)\|,$$

which implies that

$$\lim_{t \to w} |y_\infty(t)| = \infty,$$

i.e.,

$$(A(t), A_k) \in \Omega_0 \Rightarrow (-A^*(t), A_k^{*-1}) \in \Omega_\infty.$$

Since equation (1) is adjoint to (7), then, analogously,

$$(A(t), A_k) \in \Omega_\infty \Leftarrow (-A^*(t), A_k^{*-1}) \in \Omega_0.$$

For the proof of Theorem 3 we shall need the following lemma.

LEMMA 2. *Let conditions (C) hold, let the matrix $\Gamma(t) \in APC([0, w), \mathbb{C}^{n \times n})$ be nonsingular for $t \in [0, w)$, $\det \Gamma(\tau_k^+) \neq 0 \quad (k \in \mathbb{N})$ and let*

$$N(t) = \int_0^t ReTrB(\tau)d\tau + \sum_{0 < \tau_k < t} \log|\det B_k|, \tag{8}$$

where $B(\tau)$ and B_k are defined by (6). Then

$$N(t) = \log|\det\Gamma(t)| - \log|\det\Gamma(0)| + M(t). \tag{9}$$

Proof. If $X(t)$ and $Y(t)$ are fundamental matrices respectively of (1) and (5), then

$$\begin{aligned} |\det X(t)| &= |\det X(0)|\exp M(t), \\ |\det Y(t)| &= |\det Y(0)|\exp N(t), \\ \det Y(t) &= \det\Gamma(t)\det X(t), \end{aligned}$$

which implies (9). ∎

Throughout the following, the notation $H > 0, \geq 0$ etc. will mean that the matrix H is positively definite, nonnegatively definite, etc.

THEOREM 3. *Let conditions (C) hold and let the Hermitian matrix $H(t) \in APC([0,w), \mathbb{C}^{n\times n})$ be nonsingular for $t \in [0,w)$ and $\det H(\tau_k^+) \neq 0$ $(k \in \mathbb{N})$.*

(a) If

$$H'(t) + A^*(t)H(t) + H(t)A(t) \leq 0 \quad \text{a.e. on } [0,w), \tag{10}$$

$$A_k^* H(\tau_k^+)A_k = H(\tau_k) \qquad (K \in \mathbb{N}) \tag{11}$$

then $x^*(t)H(t)x(t)$ is nonincreasing and $y^*(t)H^{-1}(t)y(t)$ is a nondecreasing function in $[0,w)$, where $x(t)$ and $y(t)$ are solutions respectively of (1) and (7).

(b) If (10) and (11) are valid and $H(t) > 0$ for $t \in [0,w)$, then

$$\lim_{t\to w}\left[\frac{1}{2}\log\det H(t) + M(t)\right] = -\infty \tag{12}$$

is a necessary and sufficient condition for the existence of a nontrivial solution $x_0(t)$ of equation (1) such that

$$\lim_{t\to w} x_0^*(t)H(t)x_0(t) = 0. \tag{13}$$

Condition (12) is sufficient for the existence of a solution $y_\infty(t)$ of the adjoint equation (7) such that

$$\lim_{t\to w} y_\infty^*(t)H^{-1}(t)y_\infty(t) = +\infty.$$

Proof. (a) The function $u(t) = x^*(t)H(t)x(t)$ is nonincreasing in $[0,w)$ since $u(t) \in APC([0,w),\mathbb{R})$ and by (10) and (11) satisfies the conditions

$$\begin{aligned} u'(t) &\leq 0 \qquad \text{a.e. on } [0,w), \\ u(\tau_k^+) &= u(\tau_k) \qquad (k \in \mathbb{N}). \end{aligned}$$

Since from (10) and (11) it follows that

$$\begin{aligned} [H^{-1}(t)]' - A(t)H^{-1}(t) - H^{-1}(t)A^*(t) &\geq 0 \qquad \text{a.e. on } [0,w), \\ A_k^{-1}H^{-1}(\tau_k^+)A_k^{*-1} &= H^{-1}(\tau_k) \qquad (k \in \mathbb{N}) \end{aligned}$$

then $y^*(t)H^{-1}(t)y(t)$ is nonincreasing in $[0,w)$.

(b) Let $\Gamma(t) = H^{\frac{1}{2}}(t)$ is the unique positively definite square root of $H(t)$. Then $y(t)$ is a solution of (5) if and only if $y(t) = \Gamma(t)x(t)$, where $x(t)$ is a solution of (1) and $\|y(t)\|^2 = x^*(t)H(t)x(t)$. Thus, by part (a), $0 \leq \lim\limits_{t\to w} \|y(t)\| < \infty$ exists for any solution $y(t)$ of (5) and by Corollary 1.2 (a) $(B(t), B_k) \in \Omega_0$ iff $N(t)$ defined by (8) satisfies

$$\lim_{t\to w} N(t) = -\infty.$$

But $\log|\det\Gamma(t)| = \frac{1}{2}\log\det H(t)$ and by Lemma 2

$$N(t) = \frac{1}{2}\log\det H(t) - \frac{1}{2}\log\det H(0) + M(t),$$

whence it follows that (12) is a necessary and sufficient condition for the existence of a nontrivial solution $x_0(t)$ of (1) satisfying (13).

The second part of assertion (b) follows from the fact that if the solution $x_0(t)$ of (1) satisfies (13), then there exists a solution $y_\infty(t)$ of the adjoint equation (7) such that

$$1 \equiv y_\infty^*(t)x_0(t)$$
$$= y_\infty^*(t)H^{-\frac{1}{2}}(t)H^{\frac{1}{2}}(t)x_0(t) \le [y_\infty^*(t)H^{-1}(t)y_\infty(t)]^{\frac{1}{2}}[x_0^*(t)H(t)x_0(t)]^{\frac{1}{2}},$$

where $H^{-\frac{1}{2}} = (H^{\frac{1}{2}})^{-1} = (H^{-1})^{\frac{1}{2}}$.

Let $E \in \mathbb{C}^{n\times n}$ be the unit matrix and let $\mathcal{J} \in \mathbb{C}^{2n\times 2n}$ have the form

$$\mathcal{J} = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}.$$

Suppose that equation (1) is Hamiltonian, i.e., $A(t), A_k \in \mathbb{C}^{2n\times 2n}$ and

$$A^*(t)\mathcal{J} + \mathcal{J}A(t) = 0 \qquad (t \in [0, w)), \tag{14}$$
$$A_k^*\mathcal{J} + \mathcal{J}A_k = 2\mathcal{J} \qquad (k \in \mathbb{N}), \tag{15}$$
$$(A_k - E)^2 = 0 \qquad (k \in \mathbb{N}). \tag{16}$$

From (15) and (16) it follows that $A_k^*\mathcal{J}A_k = \mathcal{J} \quad (k \in \mathbb{N})$. This, together with (14), implies that for any two solutions $x(t), y(t)$ of (1) we have

$$y^*(t)\mathcal{J}x(t) \equiv \text{ const } = y^*(0)\mathcal{J}x(0) \qquad (t \in [0, w)).$$

Moreover, $Re Tr A(t) \equiv 0, |\det A_k| \equiv 1 \quad (t \in [0, w), k \in \mathbb{N})$.

Therefore, for the Hamiltonian impulsive equations

$$M(t) \equiv 0 \qquad (t \in [0, w))$$

and condition (12) in Theorem 3 takes the form

$$\lim_{t\to w} \det H(t) = 0. \tag{17}$$

Condition (12) is equivalent to (17) as well for the following second order impulsive equation frequently met.

EXAMPLE 1. Consider the equation

$$\begin{aligned} x'' + a(t)x &= 0 \ , \ t \neq \tau_k, \\ x'^{+} &= b_k x + x', \ t = \tau_k, \end{aligned} \tag{18}$$

where $x \in \mathbb{C}^n, a(t) \in \mathbb{C}^{n\times n}, b_k \in \mathbb{C}^{n\times n}$.

We write down (18) in the form

$$\begin{aligned} z' &= A(t)z \ , \ t \neq \tau_k, \\ z^{+} &= A_k z, \ t = \tau_k, \end{aligned} \tag{19}$$

where

$$Z = \begin{bmatrix} x \\ y \end{bmatrix}, \ A(t) = \begin{bmatrix} 0 & E \\ -a(t) & 0 \end{bmatrix}, A_k = \begin{bmatrix} E & 0 \\ b_k & E \end{bmatrix}.$$

Obviously, $TrA(t) \equiv 0, \det A_k \equiv 1$ and condition (12) has the form (17).

We shall seek the Hermitian matrix $H(t)$ satisfying the conditions of Theorem 3 in the form

$$H(t) = \begin{bmatrix} S(t) & Q(t) \\ Q^*(t) & T(t) \end{bmatrix} > 0,$$

where $S^* = S \in \mathbb{C}^{n\times n}, T^* = T \in \mathbb{C}^{n\times n}, Q \in \mathbb{C}^{n\times n}, \det ST(\tau_k^+) - |\det Q(\tau_k^+)|^2 \neq 0$.

Then :

I. Condition (12) is equivalent to

$$\lim_{t\to w} [\det S(t) \det T(t) - |\det Q(t)|^2] = 0;$$

II. Condition (10) is equivalent to

$$\begin{bmatrix} S'(t) - a^*(t)Q^*(t) - Q(t)a(t), Q'(t) - a^*(t)T(t) + S(t) \\ Q'^*(t) - T(t)a(t) + S(t), T'(t) + Q(t) + Q^*(t) \end{bmatrix} \leq 0;$$

III. Condition (11) is equivalent to

$$\begin{aligned} &S(\tau_k^+) + b_k^* Q^*(\tau_k^+) + Q(\tau_k^+)b_k + b_k^* T(\tau_k)b_k = S(\tau_k), \\ &Q(\tau_k^+) + b_k^* T(\tau_k) = Q(\tau_k), \\ &T(\tau_k^+) = T(\tau_k). \end{aligned}$$

In particular, if equation (18) is without impulse effect ($b_k \equiv 0$) then necessarily

$$S(\tau_k^+) = S(\tau_k), \qquad T(\tau_k^+) = T(\tau_k), \qquad Q(\tau_k^+) = Q(\tau_k).$$

Then, if we choose $Q(t) \equiv 0$, we derive the following.

COROLLARY 3.1. *(Compare with Corollary 3.2 of [3]). Let the $n \times n$-matrix $a(t)$ be locally integrable in $[0, w)$.*

(a) Suppose that the Hermitian matrices $S(t), T(t)$ are absolutely continuous in $[0, w)$ and such that

$$\sigma(t) \leq 0, \tau(t) \leq 0, \ \sigma(t)\tau(t) - \|S(t) - T(t)a(t)\|^2 \geq 0 \text{ a.e. in } [0, w),$$

where $\sigma(t)$ and $\tau(t)$ are the greatest eigenvalues respectively of the matrices $S'(t)$ and $T'(t)$. Then

$$E(t) = x^*(t)S(t)x(t) + x'^*(t)T(t)x'(t)$$

is a nonincreasing function for any solution $x(t)$ of the equation

$$x'' + a(t)x = 0. \tag{19}$$

Moreover, if the matrices $S(t)$ and $T(t)$ are nonsingular in $[0, w)$, then the function

$$\tilde{E}(t) = y'^*(t)S^{-1}(t)y'(t) + y^*(t)T^{-1}(t)y(t)$$

is nondecreasing for any solution $y(t)$ of the equation

$$y'' + a^*(t)y = 0. \tag{20}$$

(b) If, moreover, $S(t) > 0$ and $T(t) > 0$ for $t \in [0, w)$, then

$$\lim_{t \to w} \det S(t) \det T(t) = 0 \tag{21}$$

is a necessary and sufficient condition for the existence of a solution $x(t)$ of (19) such that $\lim_{t \to w} E(t) = 0$.

In this case, (21) is a sufficient condition for the existence of a solution $y(t)$ of (20) for which $\lim_{t \to w} \tilde{E}(t) = +\infty$.

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