Pusan Kyŏngnam Math. J. 10(1994), No. 1, pp. 199–202

DIVISORIAL RELATIONSHIP BETWEEN A NOETHERIAN RING AND ITS INTEGRAL CLOSURE

CHUL JU HWANG

It is not always true that a divisorial ideal of a Noetherian domain A comes from its integral closure $\bar{A}$ while the divisorial ideals of $\bar{A}$ contract back to divisorial ideals of A [2, Proposition 4.6]. We have interest on prime ideals and we will characterize divisorial prime ideals of A which come from the integral closure $\bar{A}$ when A is root closed.

For an integral domain A and its quotient field K , the global transform A^g of A is defined to be the set $\{x \in K | xM_1M_2 \cdots M_n \subseteq A$ for some maximal ideals $M_1, \cdots, M_n$ of $A\}$. Thus $A^g = \cup\{(M_1 \cdots M_n)^{-1} | M_i \in Max(A), n \in N\}$. There is a well-known result of Matijevic about the global transform : any ring between the Noetherian ring A and A^g is also a Noetherian ring . Pertaining to the relation of divisorial ideals of the Noetherian ring A and those of its integral closure $\bar{A}$, we have a useful result of Beck [2, Lemma 4.5]: Let A be a Noetherian domain. For any subset $\{a_1, \cdots, a_n\} \subseteq \bar{A}$, we have $A : (A : \sum_i a_i A) \subseteq \bar{A} : (\bar{A} : \sum_i a_i \bar{A})$. From this, it follows that for any ideal I of $A, I_v \subseteq (I\bar{A})_v$ and

$$(I_v\bar{A})_v = (I\bar{A})_v : I_v \subseteq (I\bar{A})_v \Rightarrow I_v\bar{A} \subseteq (I\bar{A})_v \Rightarrow (I_v\bar{A})_v \subseteq (I\bar{A})_v$$
$$\Rightarrow (I_v\bar{A})_v = (I\bar{A})_v.$$

From now on A shall denote the Noetherian domain with quotient field K. An ideal P of A is said to be strong if $PP^{-1} = P$ [1]. We denote the set of strong divisorial ideals(resp., maximal divisorial ideals) of A by $D_s(A)$ (resp.,$D_m(A)$). Let P be a prime ideal of A such that $P \in D_m(A) \cap Max(A)$ and $PP^{-1} = P, i.e., P \in D_s(A) \cap Max(A)$. Then P^{-1} is a ring and since $P^{-1} = (P :_K P) \subseteq \bar{A} \cap A^g, P^{-1}$ is a Noetherian ring by Matijevic. Also note that P is an ideal of P^{-1}. For undefined terms, the readers are referred to [3,4].

Received in May 3, 1994.

LEMMA 1. *Let P be a strong ideal of A. Then $P(P^2)^{-1} = P$ if and only if $(P^2)_v = P_v$.*

Proof. $P(P^2)^{-1} = P \Leftrightarrow (P^2)^{-1} \subseteq (P : P) = P^{-1} \Leftrightarrow (P^2)^{-1} = P^{-1} \Leftrightarrow (P^2)_v = P_v$.

LEMMA 2. *Let A be a root closed domain and $P \in D_s(A) \cap Spec(A)$. Then $P(P^{-1} :_K P) = P$ if and only if $P(P^{-1} :_K P) \subseteq \sqrt{P}$ where $\sqrt{P}$ is the radical of P in the ring P^{-1}.*

Proof. ($\Longleftarrow$) Since P is an ideal of the Noetherian ring P^{-1}, for some $n > 0, (P(P^{-1} :_K P))^n \subseteq P$. Since A is root closed, $P(P^{-1} :_K P) \subseteq A$ and so $(P^{-1} :_K P) \subseteq P^{-1} \Rightarrow (P^2)^{-1} \subseteq P^{-1} \Rightarrow (P^2)_v = P$. By Lemma 1, $P(P^2)^{-1} = P$ and $P(P^{-1} :_K P) = P$.

LEMMA 3. *Let P be a prime ideal of A. If $P = Q \cap A$ for $Q \in X^1(\bar{A})$, then $(P^2)_v \neq P$ and $\cap_{n=1}^{\infty}[(P^n)_v] = (0)$.*

Proof. Suppose $(P^2)_v = P$. We have $((P^2)_v\bar{A})_v = (P\bar{A})_v$. By Beck, $(P^2\bar{A})_v = (P\bar{A})_v$ and $((P^2\bar{A})_v)_Q = ((P\bar{A})_v)_Q$. Since $\bar{A}_Q$ is a rank one discrete valuation domain, $(P^2\bar{A})_Q = (P\bar{A})_Q$. By the cancellation law in a rank one discrete valuation domain, $(P\bar{A})_Q = \bar{A}_Q$. Now $P \not\subseteq Q$ contrary to $P = Q \cap A$. The second assertion follows from the observation $(P^n)_v \subseteq P^n\bar{A}_Q = (P\bar{A}_Q)^n$ and $\cap_{n=1}^{\infty}[(P\bar{A}_Q)^n] = (0)$ since $\bar{A}_Q$ is a rank one discrete valuation domain.

THEOREM 1. *Let A be a root closed domain and $P \in D(A) \cap Max(A)$. Then $(P^2)_v \neq P$ if and only if for a $Q \in X^1(\bar{A}), P = Q \cap A$.*

Proof. ($\Longrightarrow$) case I: $PP^{-1} \neq P$. P_P is invertible and so P_P is principal. By the principal ideal theorem, height of P_P equals 1, *i.e.*, height of P equals 1. By $LO\&INC$[5], the conclusion follows. case II: $PP^{-1} = P$. In this case $P^{-1} \subseteq A^g$ and so P^{-1} is Noetherian and $P^{-1} \subseteq \bar{A}$ too. There exists a prime ideal M of P^{-1} minimal over P such that $P(P^{-1} :_K P) \not\subseteq M$ by Lemma 1 and Lemma 2. This implies that P_M is invertible. Since $(P^{-1})_M$ is a Noetherian domain, $ht(P_M) = 1$. So $ht(M) = 1$. Since $A \subseteq P^{-1} \subseteq \bar{A}$, by $INC\&LO$, there exists a $Q \in X^1(\bar{A})$ such that $M = Q \cap P^{-1}$. Now $A \not\supseteq M \cap A \supseteq P$ implies that $M \cap A = P$ since P is a maximal ideal. Finally $P = M \cap A = Q \cap P^{-1} \cap A = Q \cap A$.

($\Longleftarrow$) This implication follows from Lemma 3.

THEOREM 2. *Let A be a root closed Noetherian domain and P a prime ideal of A. Then $(P_P^2)_v \neq (P_P)_v$ if and only if $P = Q \cap A, Q \in X^1(\bar{A})$.*

Proof. Assume that $(P_P^2)_v \neq (P_P)_v$. First we show that P_P is a divisorial ideal of A_P. Otherwise $(P_P^2)_v = ((P_P)_v)^2 = A_P = (P_P)_v$ contrary to the assumption. By Theorem 1, $(P_P^2)_v \neq P_P$ implies $P_P = Q_P \cap A_P$ for a $Q_P \in X^1(\bar{A}_P)$.

Now $P = Q \cap A, Q \in X^1(\bar{A})$. Thus $(P_P^2)_v \neq (P_P)_v \Leftrightarrow$ There exists a prime ideal $Q \in X^1(\bar{A})$ such that $P = Q \cap A$ (Note that P_P is a divisorial ideal of A_P if and only if P is a divisorial ideal of A).

COROLLARY 1. *Let $(0) \neq P$ be a prime ideal of a root-closed Noetherian domain A. Then $(P_P^2)_v \neq (P_P)_v$ if and only if $\cap_{n=1}^{\infty}[(P_P^n)_v] = (0)$.*

Proof. By Theorem 2, there exists a $Q \in X^1(\bar{A})$ such that $P = Q \cap A$. Replacing A by A_P and applying Lemma 3, we get $\cap_{n=1}^{\infty}[(P^n)_v] = (0)$.

($\Longleftarrow$) Suppose $(P_P^2)_v = (P_P)_v$. Then $((P_P^n)_v(P_P))_v = ((P_P)^2)_v$. So $(P_P^{n+1})_v = (P_P^2)_v = (P_P)_v$. Thus for $k \geq 2, (P_P^k)_v = (P_P)_v$. so $(P_P)_v = \cap_{k=1}^{\infty}((P_P^k)_v) = (0)$ by the assumption. So $P = (0)$ contrary to $P \neq (0)$.

COROLLARY 2. *Let $(0) \neq P$ be a prime ideal of a root closed Noethenian domain A. Then $((P^2)_v)_P \neq (P_P)_v$ if and only if $\cap_{n=1}^{\infty}[(P_P^n)_v] = (0)$.*

We give an example of a root closed Noetherian domain A and $P \in D_S(A) \cap Max(A)$ which is not the contraction of a divisorial prime ideal of $\bar{A}$.

Example : Let $K < F$ be extension fields such that (1) K is root closed in F, *i.e.*, for any $n \in N$ and $x \in F, x^n \in K$ implies $x \in K$ (2) $[F : K] < \infty$. For example, let Q $_0$ = Q and Q $_1$ be the smallest subfield of C containg Q and all the n th roots of elements of Q. Inductively we define Q $_{k+1}$ =Q $_k(\{x | x \in$C $, x^n \in$ Q $_k$ for some $n \in$ N $\})$. Let $K = \cup_{k=0}^{\infty} Q_k$. It can be easily shown that each element of K is contained in a radical extension of Q and K is a root closed subfied of C. Let α be a root of the polynomial $x^5 - 4x + 2$ which is not solvable by radicals over Q. Put $F = K(\alpha)$. Then K is root closed in $F, K \neq$

F,and $[F : K] < \infty$. Let $A = K + (X_1, \cdots, X_n)F[[X_1, \cdots, X_n]]$, $P = (X_1, \cdots, X_n)F[[X_1, \cdots, X_n]], B = F[[X_1, \cdots, X_n]]$. Then B is a finite A-module and B is a Noetherian A-module. By Eaken's theorem[6], A is a Noetherian ring ; A is root closed in B and $B = \bar{A}$. So A is root closed in its quotient field. Now $(A : P) \subseteq (B : P) = B$ and clearly $B \subseteq (A : P)$. Thus $P^{-1} = B \neq A$. Hence $P \subseteq P_v \not\subseteq A$. We deduce that $P = P_v$ and $P \in D_S(A) \cap Max(A)$. Since P is the contraction of the maximal ideal P of B, for $n \geq 2, P$ can not be the contraction of a height 1 prime ideal of B by INC.

References

1. V.Barucci, *Strongly divisorial ideals and complete integral closure of an integral domain*, J.Algebra **99(1)** (1986), 132–142.
2. R.Fossum, *The Divisor Class Group of a Krull Domain*, Springer Verlag, 1973.
3. R.Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, 1972.
4. J.Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, 1988.
5. I.Kaplansky, *Commutative Rings*, Allyn & Bacon, Boston, 1970
6. Matsumura, *Commutative Algebra publ Benjamin*, 1970.

Department of Mathematics
Pusan Women's University
Pusan 616-060, Korea