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ON THE SOLUTIONS OF NONLNEAR FUNCTIONAL EVOLUTION EQUATIONS IN A BANACH SPACE WITH UNIFORMLY CONVEX DUAL

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1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$. Let PC be the space of piecewise continuous functions $\psi : [-r, 0] \longrightarrow X$ for a fixed $r > 0$. PC is a Banach space with norm $\|\psi\|_{PC} = \sup\{\|\psi(\theta)\| \mid \theta \in [-r, 0]\}$ for $\psi \in PC$. In this paper, we consider the abstract nonlinear functional evolution equation

$$(FDE;\phi)\ \begin{cases} u'(t) = A(t)u(t) + G(t, u_t), & t \in [0, T], \\ u_0 = \phi, \quad \phi \in PC. \end{cases}$$

where $u : [-r, T] \longrightarrow X$ is an unknown function, $\{A(t) : D \subset X \longrightarrow X \mid t \in [0, T]\}$ is a given family of single-valued operators on X, D independent of t, $G : [0, T] \times PC \longrightarrow X$, and $\phi : [-r, 0] \longrightarrow X$ is Lipschitzian with $\phi(0) \in D$. The symbol u_t denotes the function $u_t(\theta) = u(t + \theta)$, $\theta \in [-r, 0]$. The purpose of this paper is to establish the existence of the limit solution of $(FDE;\phi)$. To do this, we give a convergence theorem for solutions of a certain approximate difference equation associated with $(FDE;\phi)$. The limits of solutions of the approximate difference equation are regarded as generalized solution of $(FDE;\phi)$ and we shall call them *limit solutions* of $(FDE;\phi)$.

Now we make further assumption regarding $A(t)$ and G.

(A.1) The dual X^* of X is uniformly convex.

(A.2) There exists $\omega \geq 0$ and a nondecreasing function $L : [0, \infty) \longrightarrow [0, \infty)$ such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle_s \leq \omega \|x_1 - x_2\|^2 + |t - s| L(\|x_2\|) \|x_1 - x_2\|$$

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for all $0 \le s \le t \le T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(A.3) For each $t \in [0,T]$ and $0 < \lambda < \lambda_0$, $\lambda_0\omega < 1$, $R(I - \lambda A(t)) = X$.

(A.4) There exists a constant $\beta > 0$ such that, for $\phi, \psi \in PC$ and $t \in [0,T]$,

$$\|G(t,\phi) - G(t,\psi)\| \le \beta\|\phi - \psi\|_{PC}.$$

(A.5) There exists $L_1 : [0,\infty) \longrightarrow [0,\infty)$ nondecreasing such that

$$\|G(t,\phi) - G(s,\phi)\| \le L_1(\|\phi\|_{PC})|t-s| \text{ for } \phi \in PC.$$

2. Preliminaries

Ley X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual space of X with $\|\cdot\|$ also denoting the norm of X^*. The value of $x^* \in X^*$ at x will be denoted by (x, x^*). Recall that the definition of the duality mapping $F : X \longrightarrow X^*$ of X, i.e., $F(x) = \{x^* \mid (x,x^*) = \|x\|^2 = \|x^*\|^2\}$. Using the Hahn-Banach theorem it is immediately clear that $F(x)$ is nonempty for any $x \in X$. In general, F is a multi-valued operator. One would need somewhat stronger condition to ensure that F is continuous. A convenient sufficient condition is given by the following.

Theorem 2.1 [1]. *If X^* is uniformly convex, then F is single-valued and is uniformly continuous on any bounded set of X. In other words, for each $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ such that $\|x\| \le M$ and $\|x - y\| < \delta$ imply $\|F(x) - F(y)\| < \varepsilon$.*

The properties of F are related to the differentiability of the norm $\|\cdot\|$ in X. For $x,\ y \in X$ and $h \in R$, let $\langle x,y\rangle_h = h^{-1}(\|x+hy\| - \|x\|)$ be the difference quotient of $\|x\|$ at x in the direction y. Since the function $h \mapsto \|x+hy\|$ is convex, we easily deduce that$h \mapsto \langle x,y\rangle_h$ is monotone increasing for $h > 0$ and$\langle x,y\rangle_h \ge -\|y\|$ for all $h > 0$. This implies the existence of the right derivative

$$\langle x,y\rangle_+ = \lim_{h\to 0+} \langle x,y\rangle_h$$

of $\|x+hy\|$ at $h=0$. As $\langle x,y\rangle_{-h} = -\langle x,-y\rangle_h$ we deduce that$\langle x,y\rangle_h$ is also monotone increasing and bounded above for$h < 0$. Thus the left derivative

$$\langle x,y\rangle_- = \lim_{h\to 0^-} \langle x,y\rangle_h$$

exists and we have $\langle x, y\rangle_- = -\langle x, -y\rangle_+$. Finally, we obtain the following inequality (see [1])

$$\langle x, y\rangle_{-h} \leq \langle x, -y\rangle_- \leq \langle x, y\rangle_+ \leq \langle x, y\rangle_h, \ h > 0.$$

For $x, y \in X$, we define the functionals $\langle\ ,\ \rangle_s$ and $\langle\ ,\ \rangle_i$ on $X \times X$ by

$$\langle y, x\rangle_s = \sup\{\langle y, x^*\rangle \mid x^* \in F(x)\}$$

and

$$\langle y, x\rangle_i = \inf\{\langle y, x^*\rangle \mid x^* \in F(x)\}.$$

Clearly $\langle y, x\rangle_s = -\langle -y, x\rangle_i = -\langle y, -x\rangle_i$ for all $x,\ y \in X$.

Definition 2.1. An operator A on X is said to be *dissipative* if for every $x_1,\ x_2 \in D(A)$ there is a $x^* \in F(x_1 - x_2)$ such that

$$\langle y_1 - y_2, x^*\rangle \leq 0, \text{ for all } y_j \in Ax_j,\ j = 1,\ 2.$$

An operator A is said to be *accretive* if $-A$ is dissipative.

Definition 2.2. Let ω be a real number.An operator A on X is said to be *ω-dissipative* if $A - \omega I$ is dissipative.

The condition (A.2) implies that, for each $t \in [0, T]$, $A(t)$ is ω-dissipative.

The following proposition is useful for later argument.

Proposition 2.1. *(1) The condition (A.2) is equivalent to the statement*

$$(1 - \lambda\omega)\|x_1 - x_2\| \leq \|x_1 - x_2 - \lambda(y_1 - y_2)\| + \lambda|t - s|L(\|x_2\|) \tag{2.1}$$

for all $\lambda > 0$, $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(2) The inequality (2.1) implies

$$\begin{aligned}(\lambda + \mu - \lambda\mu\omega)\|x_1 - x_2\| \leq & \lambda\|x_2 - \mu y_2 - x_1\| + \mu\|x_1 - \lambda y_1 - x_2\| \\ & + \lambda\mu|t - s|L(\|x_2\|)\end{aligned} \tag{2.2}$$

for all $\lambda > 0$, $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(3) The inequality (2.2) implies

$$(1 - \lambda\omega)\|x_1 - u\| \leq \|x_1 - \lambda y_1 - u\| + \lambda|A(s)u| + \lambda|t - s|L(\|u\|)$$

for all $\lambda > 0$, $0 \le s \le t \le T$, $[x_1, y_1] \in A(t)$ *and* $u \in D(A(s))$, *where* $|A(s)u| = \inf\{\|v\| \mid v \in A(s)u\}$.

3. Main result

In what follows, we assume that the conditions (A.1)-(A.5) hold. As in [7], we have the following lemmas.

Lemma 3.1. *Let* $\{t_j^n\}$ *be a partition of the interval [0,T], where* $t_j^n = jh_n = jT/n$, $j = 1, 2, \cdots, n$. *If* $n > (\omega + \beta)T$, *then there exists* $\{z_j^n\}_{j=0}^n$ *such that for* $j = 1, 2, \cdots, n$,

$$\frac{z_j^n - z_{j-1}^n}{h_n} = A(t_{j-1}^n)z_j^n + G(t_j^n, \bar{z}_{j,t_j^n}^n), \tag{3.1}$$

where

$$\bar{z}_j^n(t) = \begin{cases} \phi(t), \ t \in [-r, 0] \\ z_k^n, \ t \in (t_{k-1}^n, t_k^n] \ for \ k = 1, 2, \cdots, j-1 \\ z_j^n, \ t \in (t_{j-1}^n, T]. \end{cases}$$

and

$$\bar{z}_{j,t_{j-1}{}^n}^n(\theta) = \bar{z}_j^n(t_{j-1}{}^n + \theta), \theta \in [-r, 0]$$

Lemma 3.2. *There is a constant* $M_0 = M_0(\phi)$ *such that*

$$\sup\{\max \|z_j^n\| \mid 1 \le j \le n, \ n > (\beta + \omega)T\} \le M_0. \tag{3.2}$$

Lemma 3.3. *There exists a constant* $M_1 = M_1(\phi)$ *and an integer* $N = N(\phi)$ *such that*

$$\sup\{\max \|z_j^n - z_{j-1}^n\|/h_n \mid 1 \le j \le n, \ n \ge N\} \le M_1. \tag{3.3}$$

We now define the functions

$$z_n(t) = \begin{cases} \phi(t), \ t \in [-r, 0], \\ z_{j-1}^n + (t - t_{j-1}^n)(z_j^n - z_{j-1}^n)/h_n, t \in (t_{j-1}^n, t_j^n] \\ \qquad \text{for} j = 1, 2, \cdots, n. \end{cases} \tag{3.4}$$

Then the sequence $\{z_n(t)\}_{t\in[-r,T]}$ is uniformly Lipschitz with Lipschitz constant $M_2 = \max\{M_1, L_0\}$. Let

$$u_n(t) = \bar{z}_n^n(t),\ t \in [-r, T] \tag{3.5}$$

where $\bar{z}_n^n(t)$ is obtained from Lemma3.1(the function $u_n(t) = \bar{z}_n^n(t)$ is said to be h_n-approximate solution of $(FDE;\phi)$. Explicitly, we have

$$\bar{z}_n^n(t) = \begin{cases} \phi(t),\ t \in [-r, 0], \\ z_k^n,\ t \in (t_{k-1}^n, t_k^n] \text{ for} k = 1, 2, \cdots, n-1, \\ z_n^n = (1 - h_n A(t_{n-1}^n))^{-1}(z_{n-1}^n + h_n G(t_{n-1}^n, \bar{z}_{n,t_{n-1}^n}^n)), \\ \qquad t \in (t_{n-1}^n, T]. \end{cases}$$

The operators $A(t)$ and $G_n(t)$ are defined by

$$\begin{aligned} A_n(0) &= A(0)\phi(0), \\ A_n(t) &= A(t_{j-1}^n)z_j^n, \text{for } t_{j-1}^n < t \le t_j^n, \\ G_n(t) &= G(t_{j-1}^n, \bar{z}_{j,t_{j-1}^n}^n), \text{for } t_{j-1}^n < t \le t_j^n. \end{aligned}$$

It is easy to check that the function $z_n(t)$ is strongly differentiable on $[0, T]$ except at a finite number of points at which the strong left derivative $(d^-/dt)z_n(t)$ exists. Thus, from (3.1) and (3.4), we obtain

$$(d^-/dt)z_n(t) = A_n(t) + G_n(t),\ t \in (t_{j-1}^n, t_j^n]. \tag{3.6}$$

We will show that $u_n(t) - z_n(t) \to 0$ as $n \to \infty$ uniformly on $[-r, T]$. For $t \in (0, T]$, $t \in (t_{j-1}^n, t_j^n]$ for some $j = 1, 2, \cdots, n$. Then, from the definition of $z_n(t)$, we have

(3.7)
$$\begin{aligned} \|u_n(t) - z_n(t)\| &= \|z_j^n - z_{j-1}^n - (t - t_{j-1}^n)(z_j^n - z_{j-1}^n)/h_n\| \\ &= \|(h_n - t + t_{j-1}^n)(z_j^n - z_{j-1}^n)/h_n\| \\ &\le (t_j^n - t)M_1 \\ &\le M_1 h_n \\ &\le M_2 h_n. \end{aligned}$$

Since $u_n(t) = z_n(t) = \phi(t)$ for $t \in [-r, 0]$, we have $\|u_n(t) - z_n(t)\| \le M_2 h_n$ for $t \in [-r, T]$. On the other hand, by the Lipschitz continuity of $z_n(t)$, we have, for $t, s \in [-r, T]$,

$$\begin{aligned}
\|u_n(t) - u_n(s)\| &\le \|u_n(t) - z_n(t)\| + \|z_n(t) - z_n(s)\| \\
&\quad + \|z_n(s) - u_n(s)\| \\
&\le M_2 h_n + M_2|t - s| + M_2 h_n \\
&\le 2M_2(|t - s| + h_n) \\
&= M(|t - s| + h_n), \qquad (3.8)
\end{aligned}$$

where $M = 2M_2$.

Theorem 4.1. *The sequence $\{u_n(t)\}$ of functions converges uniformly, as $n \to \infty$, to an absolutely continuous function $u(t)$ on $[0, T]$.*

Proof. We will show that $z_n(t)$ convergesuniformly, as $n \to \infty$, to a function $u(t)$. Then it follows from above that $u_n(t)$ also converges uniformly to $u(t)$. Let $\{t_j^n\}$ and $\{t_k^m\}$ be two partitions of $[0, T]$, where $t_j^n = jh_n = jT/n$, $j = 1, 2, \cdots, n$, $t_k^n = kh_m = kT/m$, $k = 1, 2, \cdots, m$. Let $t \in (t_{k-1}^m, t_k^m] \cap (t_{j-1}^n, t_j^n]$. By the Lipschitz continuity of $z_n(t)$ and Lemma 3.1 of Kato [4], we have

(3.9)

$$\begin{aligned}
(d^-/dt)\|z_m(t) - z_n(t)\|^2 &= 2\langle (d^-/dt)z_m(t) \\
&\qquad - (d^-/dt)z_n(t), F(z_m(t) - z_n(t))\rangle \\
&= 2\langle G_m(t) - G_n(t) + A_m(t) - A_n(t), \\
&\qquad F(z_m(t) - z_n(t))\rangle \\
&\le 2\|G_m(t) - G_n(t)\|\|z_m(t) - z_n(t)\| \\
&\quad + 2\langle A_m(t) - A_n(t), F(z_m(t) - z_n(t))\rangle.
\end{aligned}$$

We also have

$$\begin{aligned}
\|G_m(t) - G_n(t)\| &= \|G(t_{k-1}^m, \bar{z}_{k,t_{k-1}^m}^m) - G(t_{j-1}^n, \bar{z}_{j,t_{j-1}^n}^n)\| \\
&\le \|G(t_{k-1}^m, \bar{z}_{k,t_{k-1}^m}^m) - G(t_{j-1}^n, \bar{z}_{k,t_{k-1}^m}^m)\| \\
&+ \ \|G(t_{j-1}^n, \bar{z}_{k,t_{k-1}^m}^m) - G(t_{j-1}^n, \bar{z}_{k,t_{j-1}^n}^n)\| \\
&+ \ \|G(t_{j-1}^n, \bar{z}_{k,t_{j-1}^n}^n) - G(t_{j-1}^n, \bar{z}_{j,t_{j-1}^n}^n)\|,
\end{aligned}$$

$$\|G(t_{k-1}^m, \bar{z}_{k,t_{k-1}^m}^m) - G(t_{j-1}^n, \bar{z}_{k,t_{k-1}^m}^m)\| \le |t_{k-1}^m - t_{j-1}^n| L_1(\|\bar{z}_{k,t_{k-1}^m}^m\|),$$

$$\begin{aligned}
&\|G(t_{j-1}^n, \bar{z}_{k,t_{j-1}^n}^m) - G(t_{j-1}^n, \bar{z}_{j,t_{j-1}^n}^n)\| \\
&\le \beta \|\bar{z}_{k,t_{j-1}^n}^m - \bar{z}_{j,t_{j-1}^n}^n\|_{PC} \\
&= \beta \sup\{\|\bar{z}_k^m(s) - \bar{z}_j^n(s)\| \\
&\qquad | \, s \in [t_{j-1}^n - r, t_{j-1}^n]\} \; . \\
&\le \beta \sup\{\|\bar{z}_k^m(s) - \bar{z}_j^n(s)\| \mid s \in [-r, t]\},
\end{aligned}$$

and

$$\|G(t_{j-1}^n, \bar{z}_{k,t_{k-1}^m}^m) - G(t_{j-1}^n, \bar{z}_{k,t_{j-1}^n}^m)\| \le \beta \|\bar{z}_{k,t_{k-1}^m}^m - \bar{z}_{k,t_{j-1}^n}^m)\|_{PC}.$$

If $t_{j-1}^n < t_{k-1}^m$, then

$$\bar{z}_k^m(t_{k-1}^m + \theta) = \bar{z}_m^m(t_{k-1}^m + \theta)$$

and

$$\bar{z}_k^m(t_{j-1}^n + \theta) = \bar{z}_m^m(t_{j-1}^n + \theta)$$

for any $\theta \in [-r, 0]$. Also, since $u_m(t) - z_m(t) \to 0$ uniformly on $[-r, T]$, there exists a sequence of positive numbers ε_m such that $\varepsilon_m \to 0$ as $m \to \infty$ and

$$\begin{aligned}
\|\bar{z}_k^m(t_{k-1}^m + \theta) - \bar{z}_k^m(t_{j-1}^n + \theta)\| &= \|\bar{z}_m^m(t_{k-1}^m + \theta) - \bar{z}_m^m(t_{j-1}^n + \theta)\| \\
&= \|u_m(t_{k-1}^m + \theta) - u_m(t_{j-1}^n + \theta)\| \\
&\le \|z_m(t_{k-1}^m + \theta) - z_m(t_{j-1}^n + \theta)\| + \varepsilon_m \\
&\le M_2 |t_{k-1}^m - t_{j-1}^n| + \varepsilon_m
\end{aligned}$$

by the Lipschitz continuity of $z_m(t)$ on $[-r, T]$. Now, it is easy to prove that the sequence $\{t_{k-1}^m - t_{j-1}^n\}$ converges to zero uniformly in j, k. From this fact, it follows that

$$\|\bar{z}_{k,t_{k-1}^m}^m - \bar{z}_{k,t_{j-1}^n}^m\| \le \bar{\varepsilon}_{m,n}$$

where $\bar{\varepsilon}_{m,n} \to 0$ as $m, n \to \infty$. Since similar inequality holds if $t^m_{k-1} < t^n_{j-1}$, we conclude that there exist sequences $\varepsilon'_{m,n}$ and $\varepsilon''_{m,n}$ such that $\varepsilon'_{m,n} \to 0$ and $\varepsilon''_{m,n} \to 0$ as $m, n \to \infty$, and

$$\begin{aligned}
&\|G_m(t) - G_n(t)\| \\
&\quad \le L_1(M_0)|t^m_{k-1} - t^n_{j-1}| \\
&\quad + \ \beta \sup\{\|\bar{z}^m_k(s) - \bar{z}^n_j(s)\| \mid s \in [-r, t]\} + \beta\varepsilon'_{m,n} \\
&\quad \le (L_1(M_0) + \beta)\varepsilon''_{m,n} + \ \beta \sup\{\|\bar{z}^m_k(s) - \bar{z}^n_j(s)\| \mid s \in [-r, t]\}
\end{aligned}$$

Now, for any $s \in [-r, T]$, we have

$$\begin{aligned}
\|\bar{z}^m_k(s) - \bar{z}^n_j(s)\| &= \|\bar{z}^m_m(s) - \bar{z}^n_n(s)\| \\
&= \|u_m(s) - u_n(s)\| \\
&\le \|z_m(s) - z_n(s)\| + \|\varepsilon^*_{m,n}(s)\|,
\end{aligned}$$

where $\varepsilon^*_{m,n}(s) \to 0$ uniformly on $[-r, T]$ as $m, n \to \infty$. Thus

$$\begin{aligned}
&\sup\{\|\bar{z}^m_k(s) - \bar{z}^n_j(s)\| \mid s \in [-r, T]\} \\
&\quad \le \sup\{\|z_m(s) - z_n(s)\| \mid s \in [-r,]\} + \varepsilon^*_{m,n},
\end{aligned}$$

where the constants $\varepsilon^*_{m,n} \to 0$ as $m, n \to \infty$. Applying the above bound to (3.9), we arrive at

(3.10)

$$\begin{aligned}
&(d^-/dt)\|z_m(t) - z_n(t)\|^2 \\
&\quad \le 2[\varepsilon^{**}_{m,n} + \beta \sup\{\|z_m(s) - z_n(s)\| \mid s \in [-r, t]\}]\|z_m(t) - z_n(t)\| \\
&\quad + \ 2\langle A_m(t) - A_n(t), F(z_m(t) - z_n(t))\rangle,
\end{aligned}$$

where $\varepsilon^{**}_{m,n} \to 0$ as $m, n \to \infty$. Using the uniform continuity of F on bounded subsets of X, we obtain a sequence of functions $\varepsilon'_{m,n}(t)$ with values in X^* such that $\lim_{m,n\to\infty} \|\varepsilon'_{m,n}(t)\| = 0$ uniformly on $[-r, T]$ and

$$F(z_m(t) - z_n(t)) = F(u_m(t) - u_n(t)) + \varepsilon'_{m,n}(t).$$

Thus

$$\begin{aligned}
&\langle A_m(t) - A_n(t), F(z_m(t) - z_n(t))\rangle \\
&= \langle A_m(t) - A_n(t), F(u_m(t) - u_n(t))\rangle + \langle A_m(t) - A_n(t), \varepsilon'_{m,n}(t)\rangle \\
&\quad \le \langle A_m(t) - A_n(t), F(u_m(t) - u_n(t))\rangle + \ [\|A(t^m_{k-1})z^m_k\| \\
&\quad + \|A(t^n_{j-1})z^n_j\|]\|\varepsilon'_{m,n}(t)\|.
\end{aligned}$$

From equation (3.1), we obtain

$$\|A(t^n_{j-1})z^n_j\| \le M_1 + \beta M_0 + C_1 \le C_7.$$

Similarly, $\|A(t^m_{k-1})z^m_k\| \le C_7$. Therefore,

$$\begin{aligned}
&(d^-/dt)\|z_m(t) - z_n(t)\|^2 \\
&\quad \le 2[\varepsilon^{**}_{m,n} + \beta \sup\{\|z_m(s) - z_n(s)\| \mid s \in [-r,t]\}]\|z_m(t) - z_n(t)\| \\
&\quad + 2\langle A_m(t) - A_n(t), F(u_m(t) - u_n(t))\rangle + 2C_7\|\varepsilon'_{m,n}(t)\|.
\end{aligned}$$

Since $(u_m(t) - u_n(t)) - (z_m(t) - z_n(t)) \longrightarrow 0$ uniformly on $[-r,T]$, there exists a sequence $\{\hat{\varepsilon}_{m,n}\}$ of positive numbers such that $\hat{\varepsilon}_{m,n} \to 0$ as $m,n \to \infty$ and

$$\|u_m(t) - u_n(t)\| \le \|z_m(t) - z_n(t)\| + \hat{\varepsilon}_{m,n}.$$

On the other hand, by the condition (A.2), we get

$$\begin{aligned}
&\langle A_m(t) - A_n(t), F(u_m(t) - u_n(t))\rangle \\
&= \langle A(t^m_{k-1})z^m_k - A(t^n_{j-1})z^n_j, F(z^m_k - z^n_j)\rangle \\
&\le \omega\|z^m_k - z^n_j\|^2 + |t^m_{k-1} - t^n_{j-1}|L(M_3)\|z^m_k - z^n_j\| \\
&\le [\omega\|z^m_k - z^n_j\| + (h_n + h_m)L(M_3)]\|z^m_k - z^n_j\| \\
&= [\omega\|u_m(t) - u_n(t)\| + \varepsilon^0_{m,n}]\|u_m(t) - u_n(t)\|,
\end{aligned}$$

where $\varepsilon^0_{m,n} = (h_n + h_m)L(M_0)$ and M'_0 is a constant such that

$$\sup\{\max\{\|z^m_k\| \mid 1 \le k \le m\} \mid m > (\omega+\beta)T\} \le M'_0,$$

and $M_3 = \max\{M_0, M'_0\}$. Hence

$$\begin{aligned}
&\langle A_m(t) - A_n(t), F(u_m(t) - u_n(t))\rangle \\
&\quad \le (\omega\|z_m(t) - z_n(t)\| + \omega\hat{\varepsilon}_{m,n} + \varepsilon^0_{m,n})(\|z_m(t) - z_n(t)\| + \hat{\varepsilon}_{m,n}) \\
&= \omega\|z_m(t) - z_n(t)\|^2 + \varepsilon^{***}_{m,n}
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon^{***}_{m,n} &= \omega\|z_m(t) - z_n(t)\|\hat{\varepsilon}_{m,n} + (\omega\hat{\varepsilon}_{m,n} + \varepsilon^0_{m,n})\|z_m(t) - z_n(t)\| \\
&\quad + \hat{\varepsilon}_{m,n}(\omega\hat{\varepsilon}_{m,n} + \varepsilon^0_{m,n})
\end{aligned}$$

and so $\varepsilon^{***}_{m,n} \to 0$ as $m, n \to \infty$. Therefore, we obtain

$$((3.11))\quad \begin{aligned}&(d^-/dt)\|z_m(t) - z_n(t)\|^2 \\ &\quad \le \varepsilon_{m,n} + 2(\beta+\omega)\sup\{\|z_m(s) - z_n(s)\|^2 \mid s \in [-r, T]\},\end{aligned}$$

where the sequence of positive constants $\varepsilon_{m,n} \to 0$ as $m, n \to \infty$. Integrating (3.11), we obtain

$$\begin{aligned}&\|z_m(t) - z_n(t)\|^2 \\ &\quad \le \varepsilon_{m,n}T + 2(\beta+\omega)\int_0^t [\sup\{\|z_m(\tau) - z_n(\tau)\| \mid \tau \in [-r, s]\}]^2 ds,\end{aligned}$$

where we have used $z_m(0) = z_n(0) = \phi(0)$.

Since for any t_1 in the interval $[0, t]$, $t_1 \in (t^n_{j-1}, t^n_j] \cap (t^m_{k-1}, t^m_k]$ for some j, k, we have

$$\begin{aligned}&\|z_m(t_1) - z_n(t_1)\|^2 \\ &\quad \le \varepsilon_{m,n}T + 2(\beta+\omega)\int_0^{t_1} [\sup\{\|z_m(\tau) - z_n(\tau)\| \mid \tau \in [-r, s]\}]^2 ds \\ &\quad \le \varepsilon_{m,n}T + 2(\beta+\omega)\int_0^{t} [\sup\{\|z_m(\tau) - z_n(\tau)\| \mid \tau \in [-r, s]\}]^2 ds.\end{aligned}$$

We actually get

$$\begin{aligned}&\sup\{\|z_m(t) - z_n(t)\|^2 \mid \tau \in [-r, t]\} \\ &\quad \le \varepsilon_{m,n}T + 2(\beta+\omega)\int_0^{t} [\sup\{\|z_m(\tau) - z_n(\tau)\| \mid \tau \in [-r, s]\}]^2 ds.\end{aligned}$$

An application of Gronwall's inequality to the above estimate shows that the sequence $z_m(t) - z_n(t) \to 0$ as $m, n \to \infty$ uniformly on $[-r, T]$. This implies that $z_n(t) \to u(t)$ uniformly on $[-r, T]$ and hence $u_n(t) \to u(t)$ uniformly on $[-r, T]$. Finally, by (4.26), we obtain

$$\|u(t) - u(s)\| \le M|t - s| \text{ for } t,\ s \in [-r, T].$$

This shows that $u(t)$ is absolutely continuous on $[-r, T]$. This completes the proof.

We say that the function$u(t) = \lim_{n\to\infty} u_n(t)$ is a limit solution of $(FDE; \phi)$.

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