Pusan Kyŏngnam Math. J. 10(1994), No. 2, pp 397–403

ORTHOGONAL IMMERSIONS WITH CONSTANT MEAN CURVATURE

CHANGRIM JANG

1.Introduction

Let $x : M^n \longrightarrow E^m$ be an isometric immersion of an n dimensional connected manifold into the m dimensional Euclidean space. Denote by Δ the Laplacian operator of M^n. The immersion x is of Finite Type k[1] if the position vector x admits the following decomposition

$$x = x_0 + x_1 + x_2 + \cdots + x_k$$

where $x_0 \in E^m$ and $\Delta x_i = \lambda_i x_i, \lambda_i \in R$. For a k type immersion x, let $E_i (i \in \{1, 2, \ldots, k\})$ denote the subspace of E^m spanned by $x_i(p), p \in M^n$. The immersion x is said to be linearly independent if the linear subspaces $E_1, E_2, \ldots, E_k$ are linearly independent. It is said to be an orthorgonal immersion if $E_1, E_2, \ldots, E_k$ are mutually orthogonal subspaces[3,4]. It is known[4] that if x is a linearly independent (resp. an orthogonal) immersion, then x satisfies the following equation

$$\Delta x = Ax + b \tag{1.1}$$

for a constant(resp. symmetric) $m \times m$ matrix A and $b \in E^m$. And it can be easily seen that if x is an orthogonal immersion, then there exist a coordinate system of E^m with respect to which x satisfies $\Delta x = Ax$ for a diagonal matrix A. The classification problem for hypersurfaces satisfying the equation (1.1) is completely solved[4,5,9]. In [8] Th.Vlachos and Th.Hasanis classified the compact submanifolds of codimension 2 satisfying the equation (1.1) for a diagonal matrix A and $b = 0$. Recently O.Garay studied orthogonal surfaces with constant mean curvature in the 4 dimensional Euclidean space and obtain the following result:

Received December 13,1994.

THEOREM A[7]. *Let $x : M^2 \longrightarrow E^4$ be an orthogonal immersion of a connected surface M^2 into E^4, with constant mean curvature. Then it is an open part of one of the following surfaces:*

(1) a minimal surface in E^4
(2) a minimal surface in S^3
(3) a helical cylinder
(4) a flat torus $S^1(r_1) \times S^1(r_2)$ in a hypersphere $S^3(r)$.

In this paper we investigate orthogonal submanifolds of codimension 2 and obtain the following theorem.

THEOREM B. *Let $x : M^n \longrightarrow E^{n+2}$ be an orthogonal immersion of a connected n dimensional manifold into the $(n + 2)$- dimensional Euclidean space , with constant mean curvature. Then M^n is one of the followings:*

(1) a minimal submanifold of E^{n+2},
(2) a minimal hypersurface of some S^{n+1},
(3) an open part of an n dimensional sphere,
(4) an open part of a product of two spheres $S^p(r_1) \times S^{n-p}(r_2)$, $p = 1, 2, \ldots, n-1$,
(5) an open part of a product of two spheres and a linear subspace of E^{n+2}, $S^p(r_1) \times S^q(r_2) \times E^{n-p-q}$, $2 \leqslant p+q \leqslant n-1, p, q \geqslant 1$,
(6) an orthogonal null 2-type submanifold.

2.Some preliminaries

Let $x : M^n \longrightarrow E^m$ be an isometric immersion. Then the metric tensor on M^n is naturally induced from that of E^m. We use the same notation $\langle\ ,\ \rangle$ for metrics on M^n and E^m. Let $\overline{\nabla}$ and ∇ be the connections of M^n and E^m respectively. Then we have the so-called Gauss formula

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{2.1}$$

where X and Y are tangent vector fields of M^n, B denotes the second fundamental form of M^n in E^m. The following equation is well-known;

$$\Delta x = trace B = \sum_{i=1}^{n} B(e_i . e_i) \tag{2.2}$$

for a local orthonormal frame $e_1, e_2, \ldots, e_n$ of M^n. Suppose the position vector field x satisfies the equation $\Delta x = Ax$ for some diagonal $m \times m$ matrix A. Then we have the following two lemmas.

LEMMA 2.1. *The equation* $\langle Ax, x\rangle = c$ *holds for some constant* c.

Proof. Let X be an arbitrary tangent vector of M^n. Differentiating $\langle Ax, x\rangle$ in the direction X, we get

$$X\langle Ax, x\rangle = 2\langle Ax, X\rangle = 0.$$

LEMMA2.2. *For a local orthonormal frame* $e_1, e_2, \ldots, e_n$ *of* M^n, *the following holds*

$$\sum_{i=1}^{n}\langle Ae_i, e_i\rangle = -\langle Ax, Ax\rangle.$$

Proof. Differentiating the equation $\langle Ax, e_i\rangle = 0$ in the direction e_i, we get

$$\langle Ae_i, e_i\rangle + \langle Ax, \overline{\nabla}_{e_i} e_i\rangle = 0.$$

From which and (2.1) , we have

$$\langle Ae_i, e_i\rangle + \langle Ax, B(e_i, e_i)\rangle = 0.$$

By summation and (2.2), we get the desired result.

3.Proof of TheoremB

By assumption there exist a coordinate system of E^{n+2} and an $(n+2) \times (n+2)$ diagonal matrix A with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_{n+2}$ such that

$$\Delta x = Ax.$$

Since M^n has a constant mean curvature, we have

$$\langle Ax, Ax\rangle = d \tag{3.1}$$

for some constant d. So we get $\langle A^2x, x\rangle = d$. This implies that A^2x is a normal vector field of M^n. We only consider the case M^n is fully contained in E^{n+2}. If M^n is contained in a hyperplane of E^{n+2}, then we know from [6] that M^n is a minimal submanifold of E^{n+2} or an open part of n dimensional sphere or an open part of a spherical cylinder which is a null 2-type submanifold. From Lemma 2.1 we know that $\langle Ax, x\rangle = c$ for some constant c. If the two normal vector fields Ax and A^2x are linearly dependent at every point of M^n, then we may assume that $A^2x = \mu Ax$ for some funtion μ, In this case , we have

$$d = \langle A^2x, x\rangle = \mu\langle Ax, x\rangle = \mu c. \tag{3.2}$$

If $c = 0$, then (3.2) that implies Ax is identically zero, i.e., M^n is a minimal submanifold of E^{n+2}. We proceed under the assumption $c \neq 0$. From (3.2) we get $\mu = \frac{d}{c}$ and $\lambda_i^2 x_i = \frac{d}{c}\lambda_i x_i$, where x_i is the ith coordinate funtion of M^n. It follows that $\lambda_i = 0$ or $\lambda_i = \frac{d}{c}$. This means that M^n is a minimal hypersurface of a hypersphere in E^{n+2} or a orthogonal null 2-type submanifold. Suppose that Ax and A^2x are linearly independent locally. Then Ax and A^2x span the normal space of M^n locally. By a direct calculation, we know that $\frac{Ax}{|Ax|}$ and $\frac{dA^2x-\langle A^3x,x\rangle Ax}{|dA^2x-\langle A^3x,x\rangle Ax|}$ are orthonormal normal vector fields of M^n. Let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame of M^n. Then $e_1, e_2, \ldots, e_n$, $\frac{Ax}{|Ax|}$ and $\frac{dA^2x-\langle A^3x,x\rangle Ax}{|dA^2x-\langle A^3x,x\rangle Ax|}$ consist a frame of E^{n+2} at each point of M^n at which $e_1, e_2, \ldots, e_n$ are defined. From some basic knowledge of linear algebra, we get

$$\sum_{i=1}^{n}\langle Ae_i, e_i\rangle + \langle A\frac{Ax}{|Ax|}, \frac{Ax}{|Ax|}\rangle +$$
$$\langle A\frac{dA^2x - \langle A^3x, x\rangle Ax}{|dA^2x - \langle A^3x, x\rangle Ax|}, \frac{dA^2x - \langle A^3x, x\rangle Ax}{|dA^2x - \langle A^3x, x\rangle Ax|}\rangle = traceA = \sum_{i=1}^{n+2}\lambda_i$$

From the above equation and Lemma 2.2, we get

$$-\langle Ax, Ax\rangle + \langle A\frac{Ax}{|Ax|}, \frac{Ax}{|Ax|}\rangle +$$
$$\langle A\frac{dA^2x - \langle A^3x, x\rangle Ax}{|dA^2x - \langle A^3x, x\rangle Ax|}, \frac{dA^2x - \langle A^3x, x\rangle Ax}{|dA^2x - \langle A^3x, x\rangle Ax|}\rangle = \sum_{i=1}^{n+2}\lambda_i. \tag{3.3}$$

Using (3.1), and after some calculations, we get the following equation from (3.3)

$$d\langle A^5x,x\rangle - \langle A^3x,x\rangle\langle A^4x,x\rangle = (\sum_{i=1}^{n+2}\lambda_i + d)(d\langle A^4x,x\rangle - \langle A^3x,x\rangle^2).$$

So the following holds

(3.4)
$$\langle A^3x,x\rangle(e\langle A^3x,x\rangle - \langle A^4x,x\rangle) + d\langle A^5x,x\rangle - ed\langle A^4x,x\rangle = 0,$$

where $e = \sum_{i=1}^{n+2}\lambda_i + d$. Without loss of genenality, we may assume that M^n can be expressed as a graph

$$(x_1, x_2, \dots, f(x_1,\dots,x_n), g(x_1,\dots,x_n))$$

locally. From the equation $\langle Ax,x\rangle = c$ and (3.1) we get

$$\begin{aligned} &\lambda_1x_1^2 + \dots + \lambda_nx_n^2 + \lambda_{n+1}f^2 + \lambda_{n+1}g^2 = c, \\ &\lambda_1^2x_1^2 + \dots + \lambda_n^2x_n^2 + \lambda_{n+1}^2f^2 + \lambda_{n+1}^2g^2 = d. \end{aligned} \tag{3.5}$$

If $\lambda_{n+1} = \lambda_{n+2}$, then from (3.5) the following holds

$$\lambda_1(\lambda_1 - \lambda_{n+1})x_1^2 + \dots + \lambda_n(\lambda_n - \lambda_{n+1})x_n^2 = d - \lambda_{n+1}c.$$

Since $x_1, x_2, \dots, x_n$ are arbitrary, we get $\lambda_i = 0$ or $\lambda_i = \lambda_{n+1}$ for $i = 1, 2, \dots, n$. This implies that Ax and A^2x are linearly dependent, which is a contradiction. Hence we can see that $\lambda_{n+1} \neq \lambda_{n+2}$. If either λ_{n+1} or λ_{n+2} is zero, we also get a contradiction. So we may assume $\lambda_{n+1}\lambda_{n+2} \neq 0$. We get the following from (3.5)

(3.6)
$$\begin{aligned} f^2 &= \frac{1}{\lambda_{n+1}(\lambda_{n+2}-\lambda_{n+1})}\{\lambda_{n+2}(c - \sum_{i=1}^n \lambda_i x_i^2) - (d - \sum_{i=1}^n \lambda_i^2 x_i^2)\}, \\ g^2 &= \frac{1}{\lambda_{n+2}(\lambda_{n+2}-\lambda_{n+1})}\{(d - \sum_{i=1}^n \lambda_i x_i^2) - \lambda_{n+1}(c - \sum_{i=1}^n \lambda_i^2 x_i^2)\}. \end{aligned}$$

Substituting (3.6) into (3.4), and after some computations, we get the following identity

$$(3.7) \qquad \sum_{i=1}^{n} C_i x_i^4 + \sum_{i<j} D_{ij} x_i^2 x_j^2 + \sum_{i=1}^{n} E_i x_i^2 + F = 0$$

where

$$(3.8) \quad C_i = \lambda_i^2(\lambda_i - \lambda_{n+1})^2(\lambda_i - \lambda_{n+2})^2(e - \lambda_i - \lambda_{n+1} - \lambda_{n+2}),$$

$$(3.9) \quad D_{ij} = \lambda_i\lambda_j(\lambda_i - \lambda_{n+1})(\lambda_i - \lambda_{n+2})(\lambda_j - \lambda_{n+1})(\lambda_j - \lambda_{n+2}) \cdot (2e - \lambda_i - \lambda_j - 2\lambda_{n+1} - 2\lambda_{n+2}),$$

(3.10)

$$E_i = \lambda_i(\lambda_i - \lambda_{n+1})(\lambda_i - \lambda_{n+2})[de(\lambda_{n+1} + \lambda_{n+2}) - ce\lambda_{n+1}\lambda_{n+2} + c\lambda_{n+1}\lambda_{n+2}(\lambda_{n+1} + \lambda_{n+2}) + d\{\lambda_i^2 + (\lambda_{n+1} + \lambda_{n+2})\lambda_i\} - de^2] + \lambda_i(\lambda_i - \lambda_{n+1})(\lambda_i - \lambda_{n+2})(e - \lambda_i - \lambda_{n+1} - \lambda_{n+2})f$$

for a constant number f, and

$$(3.11) \quad F = \lambda_{n+1}\lambda_{n+2}\{d^2(e - \lambda_{n+1} - \lambda_{n+2}) + c(\lambda_{n+1} + \lambda_{n+2})(d\lambda_{n+1} + d\lambda_{n+2} - ed - c)\}.$$

Since $x_1, x_2, \ldots, x_n$ are arbitrary, we have $C_i = D_{ij} = E_i = F = 0$ from(3.7). Hence from (3.8),(3.9),(3.10) and (3.11) we get $\lambda_i = \lambda_{n+1}$ or $\lambda_i = \lambda_{n+2}$ or $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. This means that M^n is null 3-type or 2-type. If M^n is null 3-type, then from (3.1) and $\langle Ax, x\rangle = c$ we can conclude that M^n is an open part of $S^p(r_1) \times S^q(r_2) \times E^{n-p-q}$. And if M^n is 2-type, then we can see that M^n is contained in $S^p(r_1) \times S^{n-p}(r_2)$.

REMARK. *Our result implies O.Garay's result. In* [2], *B-Y.Chen showed that if M^2 is a null 2-type surface with constant mean curvature in E^4, then M^2 is a helical cylinder. So one can prove O.Garay's Theorem, combining our theorem and B-Y.Chen's result.*

References

1. B-Y.Chen, *Total mean curvature and Submanifolds of finite type*, World Scientific Publisher, 1984.

2 B-Y Chen, *Null 2-type surfaces in Euclidean space*, Algebra,Analysis and Geometry 1-18, World Scientific Publisher, 1989
3. B-Y Chen, *Linearly independent, Orthogonal and equivariant immersions*, Kodai J. Math **14** (1991), 341–349
4. B-Y Chen and M Petrovic, *On spectral decomposition of Immersions of finite type*, Bull. Austral Math. Soc. **44** (1991), 117–129
5. Joonsang Park, *Geometric and Analytic Characterizations of Isoparametric submanifplds*, thesis (1992), Brandeis University.
6. O Garay, *An extension of Takahashi's theorem*, Geometriae Dedicata **34** (1990), 105–112.
7. O.Garay, *Orthogonal surfaces with constant mean curvature in the Euclidean 4 space, preprint*
8. T.Hasanis and T Vlachos, *Coordinate Finite Type submanifolds*, Geometriae Dedicata **37** (1991), 155–165
9 T Hasanis and T Vlachos, *Hypersurfaces of E^{n+1} satisfying $\Delta x = Ax + b$*, J. Austral Math Soc. **53** (1992), 377–384.

Department of Mathmatics
University of Ulsan
Ulsan, 680-749, Korea