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PURE SIMPLICITY OF A GROUP OVER ITS ENDOMORPHISM RING

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1.Introduction

Given any associative ring R, let M be a right R-module with a submodule N and let K be a left R-module. Then we may form the following sequence (not necessarily exact)

$$0 \longrightarrow N \otimes_R K \overset{i \otimes 1_K}{\longrightarrow} M \otimes_R K$$

where i is the inclusion homomorphism from N into M and 1_K is the identity homomorphim on K. If, for a given K, the sequence is exact for all N and M, then K is said to be left flat: if M and N are given and the sequence is exact for all K, then N is called a pure submodule of M. If an R-module has no nontrivial pure submodules then it is called pure simple. It is well known that an abelian group G forms a module over its endomorphism ring $E(G)$ where $E(G)$ is the set of all endomorphism of G. In this short paper, we will find a necessary and sufficient condition under that an abelian torsion group G is pure simple as $E(G)$-module.

2. Results

In order to describe this condition we require the following propositon which was proved by P.M.Cohn.

PROPOSITION. Let M be a right R-module. Then a submodule N of M is a pure submodule of M if and only if , for any finite sets of elements $m_i \in M$,$n_j \in N$ and $r_{ij} \in R(i = 1, \cdots m : j = 1, 2, \cdots, n)$, the relations $n_j = \sum m_i r_{ij}$ imply the existence of elements $a_i \in N$ such that $n_j = \sum a_i r_{ij}$.

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Proof. See theorem 2.4. of [1].

From the proposition, we can have the following examples.

EXAMPLES.

(1) Let p be prime. Clearly Z_{p^n} is a pure simple as $E(Z_{p_n})$-module. In fact every subgroup of Z_{p^n} is a cyclic subgroup generated by $\overline{p^k}$. Let ϕ_{p^k} be an endomorphism of Z_{p^n} defined by $(\bar{x})\phi_{p^k} = \overline{xp^k}$. Then $\bar{1}\phi_{p^k} = \overline{p^k}$, but $\bar{x}\phi_{p^k} \neq \overline{p^k}$ for every $\bar{x} \in < \overline{p^k} >$. Thus by proposition Z_{p^n} is pure simple.

(2) Let p and q be distinct primes. Then a direct sum of Z_p and Z_q is not pure simple because $Z_p \oplus 0$ and $0 \oplus Z_q$ are pure submodules of $Z_p \oplus Z_q$. We know that $E(Z_p \oplus Z_q) = \begin{pmatrix} E(Z_p) & 0 \\ 0 & E(Z_q) \end{pmatrix}$ If for every j, $\sum_i (\overline{a_i}, \overline{b_i}) \begin{pmatrix} \alpha_{ij} & 0 \\ 0 & \beta_{ij} \end{pmatrix} = (\overline{z_j}, \bar{0}) \in Z_p \oplus 0$, then for every j, $\sum_i \overline{b_i}\beta_{ij} = \bar{0}$ and $\sum_i \overline{a_i}\alpha_{ij} = \overline{z_j}$. Thus for every j, $\sum_i (\overline{a_i}, \bar{0}) \begin{pmatrix} \alpha_{ij} & 0 \\ 0 & \beta_{ij} \end{pmatrix} = (\overline{z_j}, \bar{0})$. Hence $Z_p \oplus 0$ is a pure submodule by proposition.

Using these examples,we can get the following theorem.

THEOREM. *Let G be an abelian torsion group. Then G is pure simple as $E(G)$-module if and only if G is a p-group.*

Proof. Let G be pure simple as $E(G)$-module. Since every abelian torsion group is isomorphic to $\oplus Gp$ where Gp is a p-group [2]. From above example (2) we know that G is a p-group because $Gp \oplus Gq$ is not pure simple if $p \neq q$. Conversely, we assume that G is a p-group. Then $G = A \oplus D$ where A is the reduced subgroup and D is the divisible subgroup of G respectively. We consider the following four cases.

Case 1. If $A = 0$, then G is a divisible p-group and isomorphic to a direct sum of copies of $Z(p^\infty)$. Let $\phi_p \in End(G)$ defined by $x\phi_p = xp$ and let B be a nontrivial subgroup of G. Then $\pi_\alpha B \neq Z(p^\infty)$ for some index α where π_α is a projection. In this case $\pi_\alpha B$ is generated by $\frac{\bar{1}}{p^k}$ for some k. there exists $x \in G$ such that $\pi_\alpha(x\phi_p) = \frac{\bar{1}}{p^k}$ that is

$x\phi_p \in B$. But there are no elements y in B such that $\pi_\alpha(y\phi_p) = \frac{\bar{1}}{p^k}$. So G is simple as $E(G)$-module.

Case 2. If $D = 0$, then G is isomorphic to $Z_{p^{i_1}} + Z_{p^{i_2}} + \cdots + Z_{p^{i_n}}$ ($i_1 \leq i_2 \leq \cdots \leq i_n$). Then example (1) shows that G is pure simple as $E(G)$-module.

Case 3. If $A \neq 0$, $D \neq 0$ and rank of D is 1, then A is isomorphic to $Z_{p^{i_1}} + Z_{p^{i_2}} + \cdots + Z_{p^{i_n}}$ ($i_1 \leq i_2 \leq \cdots \leq i_n$) and D is isomorphic to $Z(p^\infty)$. In this case we know that $E(G) = \begin{pmatrix} E(A) & Hom(A,D) \\ 0 & E(D) \end{pmatrix}$ where $(a,d)\begin{pmatrix} \alpha & \beta \\ 0 & j \end{pmatrix}$, $\alpha \in (A), \beta \in Hom(A,D), \gamma \in E(D)$. Note that $Hom(D,A) = 0$. Let $\beta \in Hom(A,D)$ define by the following

$$\begin{cases} (\bar{0},\bar{0},\cdots,\bar{1})\beta = \frac{\bar{1}}{p^{i_n}} & \text{and} \\ (a)\beta = 0 & \text{if } a \in Z_{p^{i_1}} \oplus Z_{p^{i_2}} \oplus \cdots \oplus Z_{p^{i_n-1}} \oplus 0 \end{cases}$$

and let $\phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. Then we can get the following equation

$$((\bar{0},\bar{0},...,\bar{1}),0)\phi = (0, \frac{\bar{1}}{p^{i_n}}) \in D$$

But we know that there are no elements of D such that $(0,d)\phi = (0, \frac{\bar{1}}{p^{i_n}})$. In fact we know that for every $d \in D$, $(0,d)\phi = (0,0)$.Thus $0 \oplus D$ is not pure submodule of G as $E(G)$-module. And $A \oplus 0$ is not characteristic subgroup of G. Hence we know that there are no pure submodules of G as $E(G)$-module.

Case 4. If rank of D is larger than 1, similarly we can know that $0 \oplus D$ is not pure submodule of G by Case 3.

From the above theorem, naturally we can get the following corollary.

COROLLARY. *Every abelian torsion group G is pure semisimple as $E(G)$-module.*

Proof. Since $G = \oplus Gp$ and each Gp is pure simple as $E(G)$-module we know that G is pure semisimple as $E(G)$-module

References

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