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REMARK ON q—BERNOULLI AND EULERIAN NUMBERS*

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0. Introduction

Throughout this paper $\mathbf{Z}_p$, $\mathbf{Q}_p$, $\mathbf{C}$ and $\mathbf{C}_p$ will respectively denote the ring of p–adic rational integers, the field of p–adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbf{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbf{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q–extension, q is variously considered as an indeterminate, a complex number $q \in \mathbf{C}$, or p–adic number $q \in \mathbf{C}_p$. If $q \in \mathbf{C}$, we assume $|q| < 1$. If $q \in \mathbf{C}_p$, we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$. The usual Bernoulli numbers are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1},$$

which can be written symbolically as $e^{Bt} = \frac{t}{e^t-1}$, interpolated to means B^k must be replaced by B_k. This relation can also be written $e^{(B+1)t} - e^{Bt} = t$, or if we equate powers of t,

$$B_0 = 1, \quad (B+1)^k - B^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

In the p–adic case, the numbers can be represented by

$$B_n = \int_{\mathbf{Z}_p} x^n d\mu_0(x)$$

where $\mu_0(x) = \mu_0(x + p^N \mathbf{Z}_p) = \frac{1}{p^N}$.

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In this paper, we will give a new relation on q–Bernoulli numbers and q–Euler numbers.

1. q–Eulerian numbers

For $q \in \mathbf{C}$ (or $\mathbf{C}_p$), we define the number $E_n(\rho : q)$, $n \geq 0$, for a root of unity $\rho \neq 1$, by

$$\frac{\rho}{qe^t - \rho} = \sum_{n=0}^{\infty} E_n(\rho : q)\frac{t^n}{n!}.$$

Let $\sum_{n=0}^{\infty} E_n(x, \rho : q)\frac{t^n}{n!} = \frac{\rho e^{xt}}{qe^t - \rho}$. Then $E_n(x, \rho : q) = \sum_{n=0}^{\infty} \binom{n}{\imath} E_n(\rho : q)x^{n-\imath}$, $\rho \neq 1$, with the usual convention of replacing $E^n(\rho : q)$ by $E_n(\rho : q)$.

For $\rho^k \neq 1$, $0 \leq a < k$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\zeta^k=1} E_n(\rho\zeta : q)\zeta^a \frac{t^n}{n!} \\
&= \sum_{\zeta^k=1} \frac{\zeta^{a+1}\rho}{qe^t - \rho\zeta} = \sum_{\zeta^k=1} \frac{\zeta^{a+1}}{\rho^{-1}qe^t - 1} \\
&= \frac{k\rho^{-a}q^k e^{at}}{\rho^{-k}q^k e^{kt} - 1} = \frac{\rho^k e^{at}}{(qe^t)^k - \rho k} kq^k \rho^{-a} \\
&= \sum_{n=0}^{\infty} E_n(\frac{a}{k}, \rho^k : q^k)\frac{(kt)^n}{n!} kq^k\rho^{-a} \\
&= k^{n+1}\rho^{-a}q^k \sum_{n=0}^{\infty} E_n(\frac{a}{k}, \rho^k : q^k)\frac{t^n}{n!}.
\end{aligned}$$

Therefore we obtain the following

LEMMA 1. *For $k > 1$, $\rho^k \neq 1$, $0 \leq a < k$, we have*

$$\sum_{\zeta^k=1} E_n(\rho\zeta : q)\zeta^a = k^{n+1}\rho^{-a}q^k E_n(\frac{a}{k}, \rho^k : q^k).$$

It is easy to see that

$$\sum_{\zeta^k=1} E_n(\rho\zeta : q) = k^{n+1}E_n(\rho^k : q^k), n \geq 0,$$

for ρ any root of unity and integer $k \geq 1$.

For $E_n(\rho : q)$, $n \geq 0$, $q \in \mathbf{C}_p$,

$$\rho = qe^{(E(\rho.q)+1)t} - \rho e^{E(\rho \cdot q)t},$$

or if we equate powers of t,

$$q(E(\rho : q) + 1)^n - \rho E_n(\rho : q) = \begin{cases} \rho & \text{if } n = 0 \\ 0 & \text{if } n \geq 1, \end{cases}$$

$$qE_0(\rho : q) - \rho E_0(\rho : q) = \rho.$$

Thus we have

$$E_0(\rho : q) = \frac{\rho}{q - \rho}.$$

However,

$$\begin{aligned} &qE_n(\rho : q) + q\sum_{i=1}^{n}\binom{n}{i}E_{n-i}(\rho : q) - \rho E_n(\rho : q) \\ &= q\sum_{i=0}^{n}\binom{n}{i}E_{n-i}(\rho) - \rho E_n(\rho) \\ &= q(E(\rho : q) + 1)^n - \rho E_n(\rho : q) = 0 \quad \text{for} \quad n \geq 1, \end{aligned}$$

so that

$$(q - \rho)E_n(\rho : q) = -q\sum_{i=1}^{n}\binom{n}{i}E_{n-i}(\rho : q) \quad \text{for} \quad n \geq 1.$$

If $\rho \neq 1$, then $(q - \rho)^{n+1}E_n(\rho : q)$, $n \geq 1$, are polynomials of ρ with coefficients in $\mathbf{Z}_p$.

2. q–Bernoulli numbers

For $q \in \mathbf{C}$ (or $\mathbf{C}_p$) with $|q| < 1$ or $|1-q|_p < p^{-\frac{1}{p-1}}$, we define

$$\frac{t}{qe^t - 1} = e^{\beta(q)t} \quad \text{and} \quad \frac{te^{xt}}{qe^t - 1} = e^{\beta(x \cdot q)t},$$

or we equate power of t,

$$q(\beta+1)^n - \beta_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $\beta^n(q)$ by β_n.

For any positive integer m, we have

$$\begin{aligned}
\frac{te^{xt}}{qe^t - 1} &= \sum_{i=0}^{m-1} \frac{tq^i e^{(x+i)t}}{q^m e^{mt} - 1} = \sum_{i=0}^{m-1} \frac{q^i}{m} \frac{mte^{\frac{(x+i)}{m}mt}}{q^m e^{mt} - 1} \\
&= \sum_{i=0}^{m-1} \frac{1}{m} q^i e^{\beta(\frac{(x+i)}{m} \cdot q^m)mt} \\
&= \sum_{i=0}^{m-1} \frac{1}{m} q^i \sum_{k=0}^{\infty} \beta_k(\frac{(x+i)}{m} : q^m) \frac{m^k t^k}{k!} \\
&= \sum_{k=0}^{\infty} (m^{k-1} \sum_{i=0}^{m-1} q^i \beta_k(\frac{(x+i)}{m} : q^m) \frac{t^k}{k!}.
\end{aligned}$$

Therefore we obtain the following

THEOREM 1. *For $k \geq 0$ integer, we have*

$$\beta_k(x : q) = m^{k-1} \sum_{i=0}^{m-1} q^i \beta_k(\frac{(x+i)}{m} : q^m).$$

If $q \to 1$, we have

$$B_k(x) = m^{k-1} \sum_{i=0}^{m-1} B_k(\frac{x+i}{m}).$$

Let f be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_{n} \mathbf{Z}/fp^n\mathbf{Z},$$
$$X^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} a + fp\mathbf{Z}_p,$$
$$a + fp^n\mathbf{Z}_p = \{x \in X | x \equiv a \pmod{fp^n}\}$$

where $a \in \mathbf{Z}$ lies in $0 \leq a < fp^n$.

Note that the natural map

$$\mathbf{Z}/fp^n\mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}$$

induces

$$\pi : X \longrightarrow \mathbf{Z}_p.$$

If g is a function on $\mathbf{Z}_p$, we denote by the same g the function $g \circ \pi$ on X. Namely we consider g as a function on X.

The above theorem 1 is important for the construction of the p-adic q-Bernoulli distribution.

THEOREM 2. *Let q be element in $\mathbf{C}_p$. For any positive N, k and d, let $\mu_{\beta,k} = \mu_{\beta,k\ q}$ be defined by*

$$\mu_{\beta,k}(a + dp^N\mathbf{Z}_p) = (dp^N)^{k-1} q^a \beta_k(\frac{a}{dp^N} : q^{dp^N}).$$

Then $\mu_{\beta,k}$ extends uniquely to distribution on X.

Proof. It suffice to check that

$$\sum_{i=0}^{p-1} \mu_{\beta,k}(a + idp^N + dp^{N+1}\mathbf{Z}_p) = \mu_{\beta,k}(a + dp^N\mathbf{Z}_p).$$

By the definition of $\mu_{\beta,k}$, we have

$$\begin{aligned}
&\sum_{i=0}^{p-1}\mu_{\beta,k}(a+idp^N+dp^{N+1}\mathbf{Z}_p)\\
&=(dp^N)^{k-1}\sum_{i=0}^{p-1}q^{a+idp^N}\beta_k\Big(\frac{a+idp^N}{dp^N}:q^{dp^{N+1}}\Big)\\
&=p^{k-1}\sum_{i=0}^{p-1}q^{idp^N}\beta_k\Big(\frac{\frac{a}{dp^N}+i}{p}:(q^{dp^N})^p\Big)\\
&=q^a(dp^N)^{k-1}\beta_k\Big(\frac{a}{dp^N}:q^{dp^N}\Big)\\
&=\mu_{\beta,k}(a+dp^N\mathbf{Z}_p).
\end{aligned}$$

Thus we have proved theorem 2.

Note that

$$\int_{\mathbf{Z}_p}1d\mu_{\beta,k}(x)=\mu_{\beta,k}(\mathbf{Z}_p)=\beta_k(q).$$

Do q–Bernoulli numbers $\beta_k(q)$ occur in Witt's formula type?

Let $\alpha\in X^*$, $\alpha\neq 1$, $k\geq 1$. For compact-open $U\subset X$, define by

$$\mu_{\alpha;k}(U)=\mu_{\beta,k:q}(U)-\alpha^{-k}\mu_{\beta,k;q^{1/\alpha}}(\alpha U).$$

Then $\mu_{\alpha;k}\to\mu_{\mathrm{Mazur},\alpha;k}$ as $q\to 1$, where $\mu_{\mathrm{Mazur},\alpha;k}$ is Mazur's measure.

3. Remark

Let $C(\mathbf{Z}_p,\mathbf{C}_p)$ and $UD(\mathbf{Z}_p,\mathbf{C}_p)$ denote the space of all continuous functions and the space of all uniformly differentiable functions on $\mathbf{Z}_p$ with values in $\mathbf{C}_p$.

Let C_{p^n} be the cyclic group consisting of all p^n–th roots of unity in $\mathbf{C}_{p^n}$ for all $n\geq 0$ and $\mathbf{T}_p$ be the direct limit of C_{p^n} with respect to the natural morphisms, hence $\mathbf{T}_p$ is the union of all C_{p^n} with discrete topology.

For $f \in UD(\mathbf{Z}_p, \mathbf{C}_p)$, we have an integral $I_0(f)$ with respect to use so called invariant measure μ_0;

$$I_0(f) = \int_{\mathbf{Z}_p} f(x) d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x)$$

and the Fourier transform $\hat{f}_q = I_0(f\phi_q)$, where ϕ_q denotes a uniformly differentiable function on $\mathbf{Z}_p$ belonging to $q \in \mathbf{C}_p$ defined by $\phi_q(x) = q^x$.

Here, we have q–analogue of Witt's formula

$$I_0(x^n \phi_q(x)) = \beta_n(q) \quad \text{for} \quad q \in \mathbf{T}_p, n \geq 0.$$

By definition, we see that

$$\frac{t}{qe^t - 1} = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{(m+1)H^m(q^{-1})}{(q-1)} t^{m+1},$$

where $H^m(q^{-1})$ means the m–th Euler numbers.

If $m \geq 1$ and $q \neq 1$, then we have

$$\beta_m = I_0(x^m \phi_q(x)) = \frac{m}{\omega - 1} H^{m-1}(q^{-1}).$$

THEOREM 3. *For $m \geq 1$, $q \in \mathbf{T}_p$, we have*

(1) $I_0(\phi_q(x)x^m) = \beta_m(q)$

(2) $\frac{\beta_m(q)}{m} = \frac{1}{q-1} H^{m-1}(q^{-1})$ *if* $q \neq 1$

(3)

$$x^n = \beta_n(1) + \sum_{\substack{q \in \mathbf{T}_p \\ q \neq 1}} \frac{\beta_n(q)}{n} \phi_q(x).$$

Now, we define the convolution for any f, $g \in UD(\mathbf{Z}_p, \mathbf{C}_p)$ due to Woodcock as follows;

$$f * g(x) = \sum_q \hat{f}_q \hat{g}_q \phi_{q^{-1}}(x).$$

Then we have $f * g \in UD(\mathbf{Z}_p, \mathbf{C}_p)$ and $(\widehat{f * g})_q = \hat{f}_q \hat{g}_q$. Another convolution $\otimes$ is induced by $*$ above $f' \otimes g' = -(f * g)'$ for $f, g \in UD(\mathbf{Z}_p, \mathbf{C}_p)$.

It is known in [3] that

$$(f \otimes g)' = f \otimes g' + f' \otimes g + f * g$$
$$f \otimes g(z) = I_0^{(x)}(f(x)g(z-x)) - f * g'(z)$$

where $I_0^{(x)}$ means the integrable with respect to the variable x.

We take $f = z^m \phi_q(z)$ and $g = z^n \phi_q(z)$.

$$\begin{aligned} &I_0(z^m \phi_q(z)) I_0(z^n \phi_q(z)) \\ &= I_0^{(z)} I_0^{(x)}(x^m q^x (z-x)^n q^{z-x}) - n I_0^{(z)}(\phi_q(z) z^m \otimes \phi_q(z) z^{n-1}). \end{aligned}$$

Let $A^q_{m,n} = I_0(\phi_q(z) z^m \otimes \phi_q(z) z^{n-1})$. Then

$$A^q_{m,n} = \frac{1}{n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j B_{m+j} \beta_{n-j}(q) - \frac{1}{n} \beta_m(q) \beta_n(q),$$

since $A^q_{m,n} = A^q_{n-1,m+1}$.

In particular, in the case of $m = 0$, $A^q_{0,n} = A^q_{n-1,1}$.

Thus we have the Euler identity, indeed, if q=1, then we have

$$-\frac{1}{n+1} \sum_{k=2}^{n-2} \binom{n}{k} B_k B_{n-k} = B_n \quad \text{for} \quad n \geq 1.$$

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