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ON THE INTEGRAL SOLUTIONS OF NONLNEAR EVOLUTION EQUATIONS IN BANACH SPACES

KUK- HYEON SON

1. Introduction

The main goal of the present paper is to study the existence of integral solution to the following Cauchy problem on a finite interval $[0,T]$:

$$(cp; x_0)\ \begin{cases} u'(t) \in A(t)u(t), & 0 \le t_0 \le t \le T, \\ u(t_0) = x_0, & x_0 \in \overline{D(A(t_0))}, \end{cases}$$

where X is a real Banach space with norm $\|\cdot\|$, $u(\cdot)$ stands for an X-valued unknown function on the interval $[0,T]$ and $\{A(t) \mid t \in [0,T]\}$ is a given family of time-dependent(possibly multi-valued) nonlinear operators acting on X with the time-dependent domain $D(A(t))$. This problem has been studied intensively in recent years, especially as regards the fundamental question of existence and uniqueness of solutions. If no additional restrictions are imposed on X, the basic method used to establish existence results has been to show, under various assumptions, the convergence of solutions of approximate difference schemes tending to $(cp; x_0)$. Recently several authors have treated the Cauchy problem $(cp; x_0)$ from the view point of difference approximation. In the autonomous the fundamental result has been established by Crandall and Liggett in [2]. An generalization of the results of Crandall and Liggett is given by Kobayashi[4]. In [4], Kobayashi introduced ω-quasi-dissipative operator and DS-limit solution of the time-indipendent(autonomous) equation

$$\begin{cases} u'(t) \in Au(t), & 0 \le t \le T \\ u(0) = x_0, & x_0 \in X. \end{cases}$$

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In [6], Pavel extended the results of Kobayashi to the time-dependent equation $(cp; x_0)$. Our purpose is to give a convergence theorem for difference approximation and to improve the results in [6].

We state our assumptions imposed on $A(t)$:

(A.1) Let ω be a real number. There exists a continuous function $f : [0,T] \longrightarrow X$ and a bounded (on bounded subsets) function $L : [0,\infty) \longrightarrow [0,\infty)$ such that

(1.1)
$$\langle y_1, x_1 - x_2\rangle_i + \langle y_2, x_2 - x_1\rangle_i \\ \leq \omega\|x_1 - x_2\|^2 + \|f(t) - f(s)\| L(\|x_2\|)\|x_1 - x_2\|$$

for all $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(A.2) The domain $D(A(t))$ of $A(t)$ depends on $t \in [t_0, T]$ in the following sense: if $t_n \to t$ in $[t_0, T]$, $x_n \in D(A(t_n))$ and $x_n \to x$ in X, then $x \in \overline{D(A(t))}$.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual space of X with $\|\cdot\|$ also denoting the norm of X^*. The value of $x^* \in X^*$ at x will be denoted by (x, x^*). Recall that the definition of the duality mapping $F : X \longrightarrow X^*$ of X, i.e., $F(x) = \{x^* \mid (x, x^*) = \|x\|^2 = \|x^*\|^2\}$. Using the Hahn-Banach theorem it is clear that $F(x)$ is nonempty for any $x \in X$. In general, F is a multi-valued operator. The properties of F are related to the differentiability of the norm $\|\cdot\|$ in X. For x, $y \in X$ and $h \in R$, let$\langle x, y\rangle_h = h^{-1}(\|x + hy\| - \|x\|)$ be the differencequotient of $\|x\|$ at x in the direction y. Since the function $h \mapsto \|x + hy\|$ is convex, we easily deduce that $h \mapsto \langle x, y\rangle_h$ is monotone increasing for $h > 0$ and $\langle x, y\rangle_h \geq -\|y\|$ for all $h > 0$. This implies the existence of the right derivative

$$\langle x, y\rangle_+ = \lim_{h \to 0+} \langle x, y\rangle_h$$

of $\|x + hy\|$ at $h = 0$. As $\langle x, y\rangle_{-h} = -\langle x, -y\rangle_h$ we deduce that $\langle x, y\rangle_h$ is also monotone increasing and bounded above for $h < 0$. Thus the left derivative

$$\langle x, y\rangle_- = \lim_{h \to 0^-} \langle x, y\rangle_h$$

exists and we have $\langle x, y\rangle_- = -\langle x, -y\rangle_+$. Finally, we obtain the following inequality (see [1])

$$\langle x, y\rangle_{-h} \leq \langle x, -y\rangle_- \leq \langle x, y\rangle_+ \leq \langle x, y\rangle_h \text{ for} h > 0.$$

For $x, y \in X$, we define the functionals $\langle\ ,\ \rangle_s$ and $\langle\ ,\ \rangle_i$ on $X \times X$ by

$$\langle y, x\rangle_s = \sup\{\langle y, x^*\rangle \mid x^* \in F(x)\}$$

and

$$\langle y, x\rangle_i = \inf\{\langle y, x^*\rangle \mid x^* \in F(x)\}.$$

Clearly $\langle y, x\rangle_s = -\langle -y, x\rangle_i = -\langle y, -x\rangle_i$ for all $x,\ y \in X$.

The following lemma is useful for later argument.

Lemma 2.1. *Let $\mathcal{F} = \{A(t) \mid t \in [0, T]\}$ be a family of nonlinear multivalued operators acting on X and ω be a real number. Then the following statements are equivalent:*

(i) $\mathcal{F}$ satisfies the condition (A.1).

(ii) For any $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$, $[x_2, y_2] \in A(s)$, $\lambda > 0$ and $\mu > 0$,

(2.1)
$$\begin{aligned}(\lambda + \mu - \lambda\mu\omega)\|x_1 - x_2\| \leq \lambda\|x_2 - x_1 - \mu y_2\| + \mu\|x_1 - x_2 - \lambda y_1\| \\ + \lambda\mu\|f(t) - f(s)\|L(\|x_2\|).\end{aligned}$$

(iii) For any $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$, $[x_2, y_2] \in A(s)$ and $\lambda > 0$,

$$\begin{aligned}(2 - \lambda\omega)\|x_1 - x_2\| \leq \|x_1 - x_2 - \lambda y_1\| + \|x_2 - x_1 - \lambda y_2\| \\ + \lambda\|f(t) - f(s)\|L(\|x_2\|).\end{aligned} \tag{2.2}$$

Each of the above statements implies

(iv) For each $[x, y] \in A(t)$, $u \in D(A(s))$, $0 \leq s \leq t \leq T$ and $\lambda > 0$,

$$(1 - \lambda\omega)\|x - u\| \leq \|x - u - \lambda y\| + \lambda|A(s)u| + \lambda\|f(t) - f(s)\|L(\|u\|), \tag{2.3}$$

where $|A(s)u| = \inf\{\|y\| \mid y \in Ax\}$.

Proof. Let us assume that (i) holds. Then (1.1) implies that there exist $x^* \in F(x_1 - x_2)$ and $y^* \in F(x_2 - x_1)$ such that

$$\langle y_1, x^*\rangle + \langle y_2, y^*\rangle \leq \omega\|x_1 - x_2\|^2 + \|f(t) - f(s)\|L(\|x_2\|)\|x_1 - x_2\|. \tag{2.4}$$

From this, we have

$$\begin{aligned}(\lambda+\mu)\|x_1-x_2\|^2 \le \lambda\langle x_1-x_2-\mu y_1, x^*\rangle \\ +\mu\langle x_2-x_1-\mu y_2, y^*\rangle+\lambda\mu\omega\|x_1-x_2\|^2 \\ +\lambda\mu\|f(t)-f(s)\|L(\|x_2\|)\|x_1-x_2\|,\end{aligned}$$

which easily gives (ii). For $\lambda=\mu$, (ii) implies (iii). If (iii) holds then we have

$$\begin{aligned}(-\lambda)^{-1}(\|x_1-x_2-\lambda y_1\|-\|x_1-x_2\|) \\ +(-\lambda)^{-1}(\|x_2-x_1-\lambda y_2\|-\|x_2-x_1\|) \\ \le \omega\|x_1-x_2\|+\|f(t)-f(s)\|L(\|x_2\|).\end{aligned}$$

Letting $\lambda\downarrow 0$ we get

$$\langle y_1, x_1-x_2\rangle_- + \langle y_2, x_2-x_1\rangle_- \le \omega\|x_1-x_2\|+\|f(t)-f(s)\|L(\|x_2\|).$$

Obviously (iii) implies (iv), and the proof is complete.

In [4], Kobayashi defined DS-approximate solutions of the problem

$$\begin{cases} u'(t)\in Au(t), & t\in(0,T), \\ u(0)=x_0, & x_0\in X, \end{cases}$$

where $A: D(A)\subset X\longrightarrow X$ is a time-independent operators acting on X with the time-independent domain $D(A)$. It is straightforward to extend this notion to the time dependent case $(cp; x_0)$.

Let t_0, $T\in R$ with $0\le t_0<T$ and $x_0\in\overline{D(A(t_0))}$. Suppose that there is a system $(\{\Delta_n\}, \{(x_k^n, y_k^n)\}, \{p_k^n\})$ of sequences with the following properties:

(i) $\{\Delta_n\}$ is a sequence of partitions of $[t_0, T]$ of the form

$$\Delta_n=\{t_0=t_0^n<t_1^n<\cdots<t_{N_n}^n=T\}\ (n\ge 1) \tag{2.5}$$

and

$$\lim_{n\to\infty}|\Delta_n|=\lim_{n\to\infty}\max\{t_k^n-t_{k-1}^n \mid 1\le k\le N_n\}=0.$$

(ii) For each $k = 1, 2, \cdots, N_n$, $x_k^n \in D(A(t_k^n))$ and $p_k^n \in X$ satisfy the difference equation

$$(2.6) \qquad y_k^n = \frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - p_k^n \in D(A(t_k^n)), \ 1 \le k \le N_n$$

as well as the following condition

$$(2.7) \qquad x_0^n \to x_0 \text{ and } b_n = \sum_{k=1}^{N_n} (t_k^n - t_{k-1}^n) \|p_k^n\| \to 0 \text{ as } n \to \infty.$$

We say that the above system $(\{\Delta_n\}, \ \{(x_k^n, y_k^n)\}, \ \{p_k^n\})$ is a *discrete scheme* for $(cp; x_0)$.

Definition 2.1. The step functions u_n on $[0, T]$ defined by

$$u_n(t) = \begin{cases} x_0^n, & \text{for } t = t_0 \\ x_k^n, & \text{for } t \in (t_{k-1}^n, t_k^n], \ k = 1, 2, \cdots, N_n \end{cases}$$

are called *DS-approximate solutions* of $(cp; x_0)$.

3. Convergence of difference approximations

In this section we treat the convergence of difference approximation of the Cauchy problem $(cp; x_0)$. Let ω be a real number. Let t_0, $\hat{t}_0 \in [0, T)$, $x_0 \in \overline{D(A(t_0))}$, $\hat{x}_0 \in \overline{D(A(\hat{t}_0))}$, and suppose that there are two discrete schemes $(\{\Delta_n\}, \{(x_k^n, y_k^n)\}, \{p_k^n\})$, $(\{\hat{\Delta}_m\}, \{(\hat{x}_k^m, \hat{y}_k^m)\}, \{\hat{p}_k^m\})$ corresponding to $(cp; x_0)$ and $(cp; \hat{x}_0)$, respectively. Namely,

(i)

$$\Delta_n = \{t_0 = t_0{}^n < t_1{}^n < \cdots t_{N_n}{}^n = T\} \ (n \ge 1)$$

and

$$\hat{\Delta}_m = \{\hat{t}_0 = \hat{t}_0^m < \hat{t}_1^m < \cdots \hat{t}_{\hat{N}_m}^m = T\} \ (m \ge 1);$$

(ii) the sequences $\{x_k^n\}$, $\{y_k^n\}$, $\{p_k^n\}$, $\{\hat{x}_j^m\}$, $\{\hat{y}_j^m\}$ and $\{\hat{p}_j^m\}$ satisfy the difference equations

$$(3.1) \qquad y_k^n = \frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - p_k^n \in D(A(t_k^n)), \ 1 \le k \le N_n$$

$$(3.2)\qquad \hat{y}_j^m = \frac{\hat{x}_j^m - \hat{x}_{j-1}^m}{\hat{t}_j^m - \hat{t}_{j-1}^m} - \hat{p}_j^m \in D(A(\hat{t}_j^m)),\ 1 \le j \le \hat{N}_m$$

for $m,\ n \geq 1$, as well as the following conditions
$x_0{}^n \to x_0, \qquad b_n = \sum_{k=1}^{N_n}(t_k{}^n - t_{k-1}{}^n)\|p_k^n\| \to 0$ and

$$(3.3)\qquad |\Delta_n| = \max\{t_k{}^n - t_{k-1}{}^n \mid 1 \le k \le N_n\} \to 0 \ \ as\ n \to \infty.$$

$\hat{x}_0^m \to \hat{x}_0, \qquad b_m = \sum_{j=1}^{\hat{N}_m}(\hat{x}_j^m - \hat{x}_{j-1}^m)\|\hat{p}_j^m\| \to 0$ and

$$(3.4)\qquad |\hat{\Delta}_m| = \max\{\hat{x}_j^m - \hat{x}_{j-1}^m \mid 1 \le j \le \hat{N}_m\} \to 0 \ \ as\ m \to \infty.$$

The DS-approximate solution $\hat{u}_m$ corresponding to the discrete scheme
$(\{\hat{\Delta}_m\}, \{(\hat{x}_k^m, \hat{y}_k^m)\}, \{\hat{p}_k^m\})$
is defined as is for u_n (see Definition 2.1), that is,

$$(3.5)\qquad \hat{u}_m(t) = \begin{cases} \hat{x}_0^m, & for\ t = \hat{t}_0 \\ \hat{x}_j^m, & for\ t \in (\hat{t}_{j-1}^m, \hat{t}_j^m]. \end{cases}$$

For simplicity of the notation, set $h_k^n = t_k^n - t_{k-1}^n$, $\hat{h}_j^m = \hat{x}_j^m - \hat{x}_{j-1}^m$ for $k = 1, 2, \cdots, N_n$ and $j = 1, 2, \cdots, \hat{N}_m$.

Then we have

$$(3.6)\ x_k{}^n - h_k{}^n y_k{}^n = x_{k-1}{}^n + h_k{}^n p_k{}^n,\ \hat{x}_j^m - \hat{h}_j^m \hat{y}_j^m = \hat{x}_{j-1}^m + \hat{h}_j^m \hat{p}_j^m$$

with $y_k^n \in A(t_k^n)x_k^n$, $\hat{y}_j^m \in A(\hat{t}_j^m)\hat{x}_j^m$ for $k = 1, 2, \cdots, N_n$ and $j = 1, 2, \cdots, \hat{N}_m$.

From now on, we drop the superscripts m and n for simplicity if there is no danger of confusion, i.e., we write t_k for t_k^n, $\hat{t}_j$ for $\hat{t}_j^m$, and so on. It is also convenient to set

$$(3.7)\qquad a_{k,j} = \|x_k - \hat{x}_j\|$$

and

$$(3.8)\ \alpha_{k,j} = \hat{h}_j/(h_k + \hat{h}_j),\ \beta_{k,j} = h_k/(h_k + \hat{h}_j),\ \gamma_{k,j} = h_k\hat{h}_j/(h_k + \hat{h}_j).$$

We now wish to estimate the difference between x_k and $\hat{x}_j$. We state a simple lemma which will be used later.

Lemma 3.1. [6] *Set, for* $0 \leq \eta < T$,

$$C_{k,j}(\eta) = [(t_k - \hat{t}_j - \eta)^2 + |\Delta_n|(t_k - t_0) + |\hat{\Delta}_m|(\hat{t}_j - \hat{t}_0)]^{1/2}, \tag{3.9}$$

Then the the inequality

$$\alpha_{k,j} C_{k-1,j}(\eta) + \beta_{k,j} C_{k,j-1}(\eta) \leq C_{k,j}(\eta) \tag{3.10}$$

holds for $k = 1, 2, \cdots, N_n$ *and* $j = 1, 2, \cdots, \hat{N}_m$.

Before proceeding to the main estimate, we need also the following lemma.

Lemma 3.2.[6] *Let* $A(t)$ *satisfy the condition (A.1) and let* u_n *be the DS-approxi-mate solution to the problem* $(cp; x_0)$. *Then for every* $r \in [0, T]$ *and* $x \in D(A(r))$, *there exists a constant* $M_0 = M_0(t_0, r, x_0, x)$, *independent of* $t \in [0, T]$ *and* $n \in N$ *such that*

$$\|u_n(t)\| \leq M_0 \tag{3.11}$$

for all $t \in [0, T]$ *and* $n \in N$.

Now, we give some remarks on the modulus of continuity of f. Set $\rho(r) = \sup\{\|f(t) - f(s)\| \mid t, s \in [0, T],\ |t - s| \leq r\}$ for $r \in [0, T]$. Obviously, $\rho : [0, T] \longrightarrow [0, \infty)$ is bounded, nondecreasing and $\lim_{r \downarrow 0} \rho(r) = 0$. Moreover, ρ is upper semicontinuous on $[0, T]$ and right semicontinuous on $[0, T)$. The simple inequality below is useful for our later proposes:

$$\rho(r) \leq \kappa^{-1} \rho(T) |r - r'| + \rho(\delta), \ r \in [0, T] \tag{3.12}$$

where $0 < \kappa < \delta \leq T$, $0 < r' < \delta - \kappa$. Let us check it: if $r \leq \delta$, then $\rho(r) \leq \rho(\delta)$, so (3.12) is trivially satisfied. If $r > \delta$ and $r' < \delta - \kappa$, we have $\kappa < \delta - r' < r - r'$, and hence $\rho(r) \leq \rho(T) \leq \dfrac{r - r'}{\kappa} \rho(T)$, thereby completing the proof of (3.12).

Using the estimates (3.10) and (3.12), we obtain the following lemma

Lemma 3.3.[6] *Let* t_0, $\hat{t}_0 \in [t_0, T)$, $x_0 \in \overline{D(A(t_0))}$, $\hat{x}_0 \in \overline{D(A(\hat{t}_0))}$, *and let* $(\{\Delta_m\}, \{(x_k^m, y_k^m)\}, \{p_k^m\})$ *and* $(\{\hat{\Delta}_n\}, \{(\hat{x}_k^n, \hat{y}_k^n)\}, \{\hat{p}_k^n\})$ *be two discrete schemes (in the sense of (3.1)-(3.4)) corresponding to* $(cp; x_0)$

and $(cp;\hat{x}_0)$, respectively. Let also the condition (A.1) be satisfied and $0 \le |\eta| < \delta < T$, $0 < \kappa < \delta - |\eta|$, and $\omega_0 = max(0,\omega)$. Assume that $|\Delta_n|$, $|\hat{\Delta}_m| < \min\{\delta - |\eta| - \kappa, 1/(2\omega_0)\}$. Then, for each $r \in [0,T]$ and every $[x,y] \in A(r)$, the following inequality holds.

(3.13)

$$\prod_{i=1}^{k}(1-\omega_0 h_i)\prod_{q=1}^{j}(1-\omega_0\hat{h}_q)\|x_k - \hat{x}_j\|$$

$$\le \|x_0^n - x\| + \|\hat{x}_0^m - x\| + C_{k,j}(t_0 - \hat{t}_0)[\|y\| + M\rho(T)]$$

$$+ \sum_{i=1}^{k} h_i\|p_i\| + \sum_{q=1}^{j}\hat{h}_q\|\hat{p}_q\|$$

$$+ M(t_0 - \hat{t}_0)[\kappa^{-1}\rho(T)C_{k,j}(\eta) + \rho(\delta)]$$

for $0 \le k \le N_n$ and $0 \le j \le \hat{N}_m$, where

$$M = \max\{L(M_0(t_0, r, x_0, x)), L(M_0(\hat{t}_0, r, \hat{x}_0, x)), L(\|x\|)\}$$

with M_0 as in (3.19),

$$L(M_0) = \sup\{L(\|y\|) \mid \|y\| \le M_0\}$$

and

$$\prod_{i=1}^{0}(1-\omega_0 h_i) = \prod_{q=1}^{0}(1-\omega_0\hat{h}_q) = 1.$$

We are now in a position to establish the convergence of DS-limit solutions.

Theorem 3.1. *Let $T > 0$, $t_0 \in [0,T)$, and $x_0 \in \overline{D(A(t_0))}$. If the family $\mathcal{F} = \{A(t) \mid t \in [0,T]\}$ satisfies the conditions (A.1) and (A.2), then the following properties holds:*

(i) There exists a continuous function $u : [0,T] \longrightarrow X$ such that any sequence u_n of DS-approximate solutions of $(cp;x_0)$ is convergent to u as $n \to \infty$, uniformly on $[t_0,T]$;

(ii) $u(t) \in \overline{D(A(t))}$ for each $t \in [0,T]$ and $u(t_0) = x_0$.

Proof. We shall use Lemma 3.3 with $t_0 = \hat{t}_0$, $x_0 = \hat{x}_0$, $x_j = \hat{x}_j$, $t_j = \hat{t}_j$, $h_j = \hat{h}_j$ and $p_j = \hat{p}_j$. Let $t \in [t_0,T)$ and let $k = k_n$ and

$j = j_m$ be such that $t \in (t_{k_n-1}, t_{k_n}] \cap (t_{j_m-1}, t_{j_m}]$. By (3.10) we see that $C_{k_n,j_m}(0) \to 0$ as $m,\ n \to \infty$, because $t_{k_n} \to t$ and $t_{j_m} \to t$ as $m,\ n \to \infty$. On the other hand,

$$\omega_{k,j}^{-1} \leq \exp[4\omega_0(T - t_0)] \equiv C. \tag{3.14}$$

By the definition of u_n it follows that $u_n(t) = x_{k_n}$ and $u_m(t) = x_{j_m}$ (hence $u_n(t) \in D(A(t_{k_n}))$). Consequently, with $\eta = 0$, (3.13) yields, for $t_0 = \hat{t}_0 = r$ and $x \in D(A(t_0))$,

(3.15)

$$\begin{aligned} \|u_n(t) - u_m(t)\| \leq C[\ & |x_0^n - u| + \|\hat{x}_0^m - u\| \\ & + C_{k_n,j_m}(0)(\|A(r)x\| + M\rho(T)) + \sum_{i=1}^{k} h_i \|p_i\| \\ & + \sum_{q=1}^{j} \hat{h}_q \|\hat{p}_q\| \\ & + M(\hat{t}_{j_m} - t_0)(\kappa^{-1}\rho(T)C_{k_n,j_m}(0) + \rho(\delta))M\ ] \end{aligned}$$

for $0 \leq k_n \leq N_n$ and $0 \leq j_m \leq \hat{N}_m$, and hence

$$\lim_{m,n\to\infty} \|u_n(t) - u_m(t)\| \leq C[2\|x_0 - x\| + M(T - t_0)\rho(T)] \tag{3.16}$$

for all $x \in D(A(t_0))$ and $\delta > 0$. Since $\lim_{\delta \downarrow 0} \rho(\delta) = 0$ and x can be taken in $D(A(t_0))$ so that $\|x_0 - x\|$ is sufficiently small, we infer that $\lim_{m,n\to\infty} (u_n(t) - u_m(t)) = 0$ uniformly with respect to $t \in [t_0, T]$. Note also that $u_n(t) \in D(A(t_{k_n}))$ and

(3.17)

$$u(t; t_0, x_0) = \lim_{n\to\infty} u_n(t) \text{ jointly } t_{k_n} \to t \text{ implies } u(t; t_0, x_0) \in \overline{D(A(t))}$$

by condition (A.2). Arguing as above (in view of (3.13)) it is clear that any other DS-approximate to $\hat{u}_n$ corresponding to $t_0 \in [0, T)$ and $x_0 \in \overline{D(A(t_0))}$ is also convergent to u. It remains to prove the continuity of u on $[t_0, T]$. To this end, take $t,\ t' \in [t_0, T]$ and $n \in N$. Let k_n and j_n be such that $t_{k_n-1} < t \leq t_{k_n}$, $t_{j_n-1} < t' \leq t_{j_n}$. Then $t_{k_n} \to t$ and $t_{j_n} \to t'$ as $n \to \infty$ and $x_{k_n} = u_n(t)$, $x_{j_n} = u_n(t')$. In this case $C_{k_n,j_n}(0) \to |t-t'|$ as $n \to \infty$. Consequently, with $t_0 = \hat{t}_0$, $t_j = \hat{t}_j$, $x_j = \hat{x}_j$, $m = n$ and $\eta = 0$, (3.13) yields

$$\begin{aligned} \|u(t)-u(t')\| &= \lim_{n\to\infty} \|u_n(t)-u_n(t')\| \\ &\le C[2\|x_0 - x\| + |t-t'|\{\|A(t_0)x\| + M\rho(T)\} \\ &\quad + M(t'-t_0)\{\kappa^{-1}\rho(T)|t-t'| + \rho(\delta)\}] \end{aligned} \tag{3.18}$$

for every $x \in D(A(t_0))$, $0 < \delta < T$ and $0 < \kappa < \delta$. Since $\lim_{\delta\downarrow 0}\rho(\delta) = 0$ and x can be chosen so that $\|x_0 - x\|$ is arbitrarily small, (3.43) implies that u is strongly uniformly continuous on $[t_0, T]$. This completes the proof.

By virtue of Theorem 3.1, we define the following.

Definition 3.2. Let $t_0 \in [0, T)$ and $x_0 \in \overline{D(A(t_0))}$. A continuous function u on $[t_0, T]$ is said to be a *DS-limit solution* of the problem $(cp; x_0)$ if there exist DS-approximate solutions u_n of this problem on $[t_0, T]$, uniformly convergent to u (on $[t_0, T]$).

4. Main result

In this section, we investigate some basic properties of DS-limit solutions of the Cauchy problem $(cp; x_0)$. Let ω be a real number and $T > 0$ be fixed.

Definition 4.1. Let $t_0 \in [0, T]$ and $x_0 \in \overline{D(A(t_0))}$. An X-valued function $u(t)$ on $[t_0, T]$ is said to be a *strong solution* of $(cp; x_0)$ on $[t_0, T]$ if the following conditions are satisfied:

(i) $u(t_0) = x_0$,

(ii) $u(t)$ is absolutely continuous on $[t_0, T]$,

(iii) $u(t)$ is differentiable a.e. on (t_0, T), $u(t) \in D(A(t))$ and satisfies the problem $(cp; x_0)$ a.e. on (t_0, T).

To get into the notion of integral solution, suppose that $u(t)$ is a strong solution of $(cp; x_0)$ under hypothesis (A.1). Take arbitrary $r \in [t_0, T]$ and $[x, y] \in A(r)$. Since $u'(t) \in A(t)u(t)$ for almost everywhere $t \in [t_0, T]$, an application of Lemma 1.3 of Kato [2] and condition (A.1) yield

(4.1)

$$(d/dt)\|u(t)-x\| = \langle u(t)-x, u'(t)\rangle_- \\ \leq \langle u(t)-x, y\rangle_+ + \omega\|u(t)-x\| \\ + \|f(t)-f(r)\|L(\|x\|) \\ \leq \omega\|u(t)-x\| + \langle u(t)-x, y\rangle_+ + K\|f(t)-f(r)\|,$$

where

(4.2) $K = \max\{L(C_1) \text{ and } L(\|x\|)\},\ C_1 \geq \sup\{\|u(t)\| \mid t_0 \leq t \leq T\}.$

Integrating (4.1) over $[t,t']$ one obtain

(4.3)

$$\|u(t')-x\| - \|u(t)-x\| \\ \leq \int_t^{t'} [\omega\|u(\tau)-x\| + \langle u(\tau)-x, y\rangle_+ + K\|f(\tau)-f(r)\|]d\tau$$

for all $t_0 \leq t \leq t' \leq T$, $r \in [t_0,T]$ and $[x,y] \in A(r)$.

Definition 4.2. By an *integral solution* of the Cauchy problem $(cp; x_0)$ on $[t_0,T]$, we mean a continuous function $u(t)$ on $[t_0,T]$ satisfying the inequality (4.3) with K as in (4.2), $u(t_0) = x_0$ and $u(t) \in \overline{D(A(t))}$ for $t \in [t_0,T]$.

Theorem 4.1. *Suppose that the family $\mathcal{F} = \{A(t) \mid t \in [0,T]\}$ satisfies the conditions (A.1) and (A.2). If u is a DS-limit solution of the problem $(cp; x_0)$, then u is the unique integral solution of this problem.*

Proof. We first prove that the DS-limit solution u is an integral solution. To accomplish this assertion, let $[x,y] \in A(r)$, $r \in [t_0,T]$ and $t,\ t' \in [t_0,T]$. Since $\{x_k\}$ is bounded (by (3.11)) there exists a constant $K \geq \max\{L(\sup\|x_k\|), L(\|x\|)\}$ such that

(4.4)

$$\langle\frac{x_k-x_{k-1}}{h_k} - p_k, x_k - x\rangle_s + \langle y, x-x_k\rangle_s \leq \omega\|x_k-x\|^2 + K\|f(t_k)-f(r)\|.$$

Since $y_k = \dfrac{x_k - x_{k-1}}{h_k} - p_k$ and $(x_k - x) - (x_{k-1} - x) = h_k(y_k + p_k)$, we have

(4.5) $$\|x_k - x\| - \|x_{k-1} - x\| \leq h_k\langle x_k - x, y_k\rangle_- + h_k\|p_k\|$$

for $1 \le k \le N_n$. Let $0 \le j \le k \le N_n$. Since

$$\langle x_k - x, y_k\rangle_- + \langle x_k - x, -y\rangle_- \le \omega\|x_k - x\| + K\|f(t_k) - f(\tau)\|$$

for $1 \le k \le N_n$, we obtain the estimate

$$\|x_k - x\| - \|x_j - x\| \le \sum_{i=j+1}^{k} h_i[\langle x_i - x, y\rangle_+ + \omega\|x_i - x\| + K\|f(t_i) - f(\tau)\| + \|p_i\|] \tag{4.6}$$

Let $k = k_n$ and $j = j_n$ be such that $t \in (t_{j_n-1}, t_{j_n}]$ and $t' \in (t_{k_n-1}, t_{k_n}]$. Also set $a_n(\tau) = t_{k_n}$ for $\tau \in (t_{k_n-1}, t_{k_n}]$. According to the definition of u, (4.6) becomes

$$\begin{aligned}\|u_n(t') - x\| - \|u_n(t) - x\| &\le \int_{t_{j_n}}^{t_{k_n}} [\omega\|u_n(a_n(\tau)) - x\| + \langle u_n(a_n(\tau)) - x, y_k\rangle_+ \\ &+ b_n + K\|f(a_n(\tau)) - f(\tau)\|]d\tau.\end{aligned} \tag{4.7}$$

Clearly $a_n(\tau) \to \tau$ as $n \to \infty$ (uniformly with respect to τ), and hence $u_n(a_n(\tau)) \to u(\tau)$ as $n \to \infty$, uniformly with respect to $\tau \in [t_0, T]$. Passing through the limit for $n \to \infty$ in (4.7), one obtain (4.3). Hence u is an integral solution of the problem $(cp; x_0)$.

To prove the uniqueness, let $\bar{u}$ be arbitrary integral solution of the problem $(cp; x_0)$ and u be the DS-limit solution to this problem. We will prove that $\bar{u} = u$ on $[t_0, T]$. Let $0 \le s \le s' \le T$. Then, substituting $\bar{u}$, s, s', and $[x_k, y_k]$ for u, t, t', and $[x, y]$, respectively, in the inequality (4.3), we have

$$\begin{aligned}\|\bar{u}(s') - x_k\| - \|\bar{u}(s) - x_k\| &\le \int_s^{s'} [\omega\|\bar{u}(\tau) - x_k\| + \langle \bar{u}(\tau) - x_k, y_k\rangle_+ \\ &+ K\|f(\tau) - f(t_k)\|]d\tau.\end{aligned} \tag{4.8}$$

Since

$$h_k\langle \bar{u}(\tau) - x_k, y_k\rangle_+ \le \|\bar{u}(\tau) - x_{k-1}\| - \|\bar{u}(\tau) - x_k\| + h_k\|p_k\|,$$

we obtains
(4.9)
$$h_k(\|\bar{u}(s') - x_k\| - \|\bar{u}(s) - x_k\|)$$
$$\leq \int_s^{s'} \omega h_k \|\bar{u}(\tau) - x_k\| d\tau$$
$$+ \int_s^{s'} (\|\bar{u}(\tau) - x_{k-1}\| - \|\bar{u}(\tau) - x_k\|) d\tau$$
$$+ K \int_s^{s'} h_k \|f(\tau) - f(t_k)\| d\tau + h_k(s' - s)\|p_k\|.$$

Integrating (4.9) for $i = j+1, \cdots, k$, we get

$$\int_{t_j}^{t_k} (\|\bar{u}(s') - u_n(\eta)\| - \|\bar{u}(s) - u_n(\eta)\|) d\eta$$
$$\leq \int_s^{s'} (\|\bar{u}(\tau) - u_n(t_j)\| - \|\bar{u}(\tau) - u_n(t_k)\| d\tau$$
$$+ \int_{t_j}^{t_k} \int_s^{s'} [\omega \|\bar{u}(\tau) - u_n(\eta)\| + K\|f(\tau) - f(a_n(\eta))\|] d\tau d\eta$$
$$+ (s' - s) \sum_{i=j+1}^{k} h_i \|p_i\|$$

holds for $0 \leq j \leq k \leq N_n$. Letting $t_k \to t'$ and $t_j \to t$ as $n \to \infty$, we have

$$\begin{aligned}\int_t^{t'} &(\|\bar{u}(s') - u(\eta)\| - \|\bar{u}(s) - u(\eta)\|)d\eta \\ &+ \int_s^{s'} (\|\bar{u}(\tau) - u(t')\| - \|\bar{u}(\tau) - u(t)\|d\tau \\ &\leq \omega \int_t^{t'} \int_s^{s'} \|\bar{u}(\tau) - u(\eta)\| d\tau d\eta \\ &+ K \int_t^{t'} \int_s^{s'} \|f(\tau) - f(\eta))\| d\tau d\eta \\ &\leq \omega_0 \int_t^{t'} \int_s^{s'} \|\bar{u}(\tau) - u(\eta)\| d\tau d\eta \\ &+ K \int_t^{t'} \int_s^{s'} \|f(\tau) - f(\eta))\| d\tau d\eta\end{aligned}$$

where $\omega_0 = \max(0, \omega)$. We now apply Proposition B of [5] (Proposition 4.1 below). Putting $[a, b] = [t_0, T]$, $\phi(t', t) = \|\bar{u}(t') - u(t)\|$, $c = \omega_0$, $d = K$ and $\psi(t', t) = \|f(t') - f(t)\|$ in the proposition below, we have

$$e^{-\omega_0 t'} \|\bar{u}(t') - u(t')\| \leq e^{-\omega_0 t} \|\bar{u}(t) - u(t)\|$$

for $t_0 \leq t \leq t' \leq T$, which implies u is the only integral solution of $(cp; x_0)$.

Proposition 4.1 [5]. *Let $a < b$ and ϕ, ψ be nonnegative functions defined on all of $[a, b] \times [a, b]$ satisfying the following conditions:*

(i) ϕ is continuous on $[a, b] \times [a, b]$,

(ii) ψ is upper semicontinuous on $[a, b] \times [a, b]$,

(iii) there exist constants c, $d > 0$ such that for $a \leq s < t \leq b$ and $a \leq \sigma \leq \tau \leq b$ we have

$$\begin{aligned}\int_\sigma^\tau &[\phi(\xi, t) - \phi(\xi, s)]d\xi + \int_s^t [\phi(\tau, \eta) - \phi(\sigma, \eta)]d\eta \\ &\leq c \int_s^t \int_\sigma^\tau \phi(\xi, \eta) d\xi d\eta + d \int_s^t \int_\sigma^\tau \psi(\xi, \eta) d\xi d\eta.\end{aligned}$$

Then, we have

$$e^{-ct}\phi(t,t) - e^{-cs}\phi(s,s) \leq de^{-cs}\int_s^t \psi(\xi,\xi)d\xi$$

for $a \leq s \leq t \leq b$. *If, in particular,* $\psi(s,s) = 0$ *for all* $s \in [a,b]$, *then*

$$e^{-ct}\phi(t,t) \leq e^{-cs}\phi(s,s)$$

for $a \leq s \leq t \leq b$.

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Department of Mathematics
Dong-Seo University
616-010 Pusan, Korea