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ON THE EXISTENCE OF SOME HOLOMORPHIC FUNCTIONS

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Let $S = \{z = x + iy | 0 \leq x \leq 1, -\infty < y < \infty\}$. Suppose that f is continuous on S. Is there any holomorphic function on S^0 with the following properties: $Ref(iy) \geq 1, Ref(1+iy) \geq 0$ and $Ref(x) < 1 - x$ for some $0 < x < 1$ and for all y in R ?

There are examples satisfying partially those conditions (see: $f(z) = 1 - z, f(z) = \cos \frac{\pi z}{2}$ or $f(z) = \tan \frac{(1-z)\pi}{4}$).

The question was by product and solved accidently when we consider some problems related to Stein's complex interpolation theorem(see [1]).

THEOREM 1.. *Let g be continuous function on S. Then there is no such a holomorphic function on S^0 satisfying those three conditions.*

Proof. Suppose that there exist such a holomorphic function g on S^0 such that $Reg(t_0) < 1 - t_0$ for some $0 < t_0 < 1$.

Let $\{\phi^j\}$ be a smooth partition of unity on $(0, \infty)$ with ϕ^1 supported in $(1,3)$ and ϕ^0 supported near 0 such that $\phi^j(r) = \phi^1(2^{1-j}r), j = 0, 1, 2, \ldots$. We consider operators R_j on R^n given by $\phi^j(|x|)\hat{f}(x) = \widehat{R_j f}(x)$. Now classical estimates on the size of the Bessel function says(see [3]): $|t^{-(a+ib)} J_{a+ib}(t)| \leq C_a e^{c|b|}(1 + t)^{-a-1/2}$ where J_α is a Bessel function of order α.

Let us consider spherical means of complex order α (extended by analytic continuation) $M^\alpha f(x) = m_\alpha * f(x)$ given by $\widehat{M^\alpha f}(x) = \widehat{m}_\alpha(x)\hat{f}(x)$ where $\widehat{m}_\alpha(x) = \pi^{-\alpha+1}|x|^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi x)$. The Plancheral theorem gives us

$$\begin{aligned}\{\int_{R^2} |M^\alpha f(x)|^2 dx\}^{1/2} &\leq \{\int_{R^2} |\widehat{m}_\alpha(x)|^2 |\hat{f}(x)|^2\}^{1/2} \\ &\leq C_\alpha e^{c|Im\alpha|}\{\int_{R^2} (1 + |x|)^{-1-2Re\alpha} |\hat{f}(x)|^2 dx\}^{1/2}\end{aligned}$$

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for $n = 2$ and $f \in \mathcal{S}$.

Let us define a family of operators $T_z^j : L^p(R^2) \longrightarrow L^p(R^2)$ given by $T_z^j f(x) = 2^{j\beta(z)} M^{\alpha(z)} R_j f(x)$ where $\alpha(z) = \alpha_0(1-z) + \alpha_1 z$ and $\beta(z) = \{\alpha_0 - (K(t_0) + \frac{1}{2})\} g(z) + \alpha_1(z)$. Set $\alpha_0 = -1$ and $\alpha_1 = \epsilon > 0$. Then

$$\begin{aligned}\|T_{iy}^j f(x)\|_2 &= 2^{jRe\beta(iy)} \|M^{\alpha(iy)} R_j f(x)\|_2 \\ &\leq 2^{j(-3/2-k(t_0))} \|M^{\alpha}(iy) R_j f(x)\|_2 \\ &\leq 2^{j(-1-k(t_0))} e^{c(\epsilon+1)|y|} \|f\|_2\end{aligned}$$

Let $M_0^j(y) = 2^{j(-1-k(t_0))} e^{c(\epsilon+1)|y|}$. Then M_0^j satisfies a growth condition: $[\sup_{-\infty<y<\infty} e^{-b|y|} \log M_0^j(y) < \infty]$ for some $b < \pi$. Also, we have

$$\begin{aligned}|T_{1+iy}^j f(x)| &= 2^{jRe\beta(1+iy)} |M^{\alpha(1+iy)} f(x)| \\ &\leq 2^{j\epsilon} \left|\frac{1}{\Gamma(\epsilon + (\epsilon+1)iy)}\right| \int_{R^2} (1-|y|^2)_+^{\epsilon-1+(\epsilon+1)iy} |R_j f(x-y)| dy \\ &\leq c 2^{j\epsilon} e^{c(\epsilon+1)|y|/2} \|f\|_\infty\end{aligned}$$

since $\|R_j f\|_\infty \leq \{\int_{R^2} |\hat{\phi}^1(|y|)| dy\} \|f\|_\infty$.

Put $M_1^j = c 2^{j\epsilon} e^{c(\epsilon+1)|y|/2}$. Then M_1^j has a polynomial growth satisfying Stein's growth condition.

By the Complex interpolation theorem between $\frac{1}{p_0} = \frac{1}{2}$ and $\frac{1}{p_1} = 0$,

$$\begin{aligned}\|T_t^j f(x)\|_p &= 2^{j\beta(t)} \|M^{\alpha(t)} R_j f(x)\|_p \\ &\leq M_t^j \|f\|_p\end{aligned}$$

for $1 - t = \frac{2}{p}, 0 < t < 1$.

It is easy to see that

$$\begin{aligned}M_t^j &= exp\{\frac{\sin \pi t}{2} \int_{-\infty}^{\infty} (\frac{\log M_0^j(y)}{\cosh \pi y - \cos \pi t} + \frac{\log M_1^j(y)}{\cosh \pi y + \cos \pi t}) dy\} \\ &\leq c(t) 2^{j(-1-k(t_0))(1-t)} 2^{j\epsilon t}.\end{aligned}$$

where c depends on t.

Let $t = t_0$. Then we obtain

$$\|M^{\alpha(t_0)} R_j f(x)\|_p \leq c 2^{-j\beta(t_0)+j(-1-k(t_0))(1-t)+j\epsilon t} \|f\|_p$$
$$= c 2^{j/2((-2-k(t_0))(1-t_0-Reg(t_0))+Reg(t_0))} \|f\|_2$$

where $1 - t_0 = \frac{2}{p}$.

We can take $k(t_0)$ as the smallest positive integer such that $k(t_0) > 2\{\frac{Ref(t_0)}{1-t_0-Ref(t_0)} - 1\}$.

Thus we get

$$\|M^\alpha f\|_p \leq \sum_{j=0}^{\infty} \|M^\alpha R_j f\|_p \leq c\|f\|_p$$

where $\frac{1}{p} = \frac{1-t_0}{2}$ and $\alpha > -1(1-t_0)$ since ϵ is arbitrary.

The complex interpolation theorem of Stein between $(0,0)$ and $(1,0)$ gives us $\|M^\alpha f\|_\infty \leq c\|f\|_p$ if $\alpha > \frac{1}{p}$. By repeating interpolation theorem between $(\frac{1}{p}, \frac{1}{p})$ and $(\frac{1}{p}, 0)$, we obtain $\|M^\alpha f\|_{3p} \leq c\|f\|_p$ for $\alpha > 0$ and $\frac{2}{p} = (1 - t_0)$ This contradicts to a necessary condition in [2].

Thus there is no such a holomorphic function.

References

1. E. Stein and G Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1970.
2. R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans Amer Math. Soc. **148** (1970), 461-471.
3. G. Watson, *Theory of Bessel functions*, Cambridge University Press,, Cambridge, 1948

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