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ON A CONJECTURE OF GRAHAM

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Let A be a finite sequence of n positive integers $a_1 < a_2 < \ldots < a_n$. Graham [2] has conjectured that $\max_{i,j}\{a_i/(a_i,a_j)\} \geq n$. The conjecture has been verified in some special cases (see references).

In this paper we verify the conjecture in three cases.

A prime p is called a *prime factor* of A if p is a prime factor of some a_i. Denote by $P(a_i)$ (resp. $P(A)$) the set of all prime factors of a_i (resp. A).

THEOREM 1. *If* $P(a_k) = \{p_1, p_2, \ldots, p_s\}$ *for some* a_k, *and* $a_i \neq p_1^{d_1}p_2^{d_2}\ldots p_s^{d_s}a_j$ *for* $i \neq j$, $d_1, d_2, \ldots, d_s \geq 0$, *then* $\max_{i,j}\{a_i/(a_i,a_j)\} \geq n$.

Proof. Let $a_i = p_1^{i_1}p_2^{i_2}\ldots p_s^{i_s}b_i$ for all a_i where $i_1, i_2, \ldots, i_s \geq 0$ and p_1, p_2, ..., $p_s \nmid b_i$. Then $b_i \neq b_j$ for $i \neq j$ and $b_k = 1$. So $\max_{i,j}\{b_i/(b_i,b_j)\} \geq n$. Since

$$\begin{aligned}
a_i/(a_i,a_j) &= (p_1^{i_1}p_2^{i_2}\ldots p_s^{i_s}b_i)/(p_1^{i_1}p_2^{i_2}\ldots p_s^{i_s}b_i, p_1^{j_1}p_2^{j_2}\ldots p_s^{j_s}b_j) \\
&= (p_1^{i_1}p_2^{i_2}\ldots p_s^{i_s}b_i)/((p_1^{i_1}p_2^{i_2}\ldots p_s^{i_s}, p_1^{j_1}p_2^{j_2}\ldots p_s^{j_s})(b_i,b_j)) \\
&\geq b_i/(b_i,b_j)
\end{aligned}$$

for all i, j, it follows that $\max_{i,j}\{a_i/(a_i,a_j)\} \geq \max_{i,j}\{b_i/(b_i,b_j)\} \geq n$.

COROLLARY 2. *Suppose that* A *contains a prime power* p^d *with the property:* $a_i \neq p^k a_j$ *for* $i \neq j$ *and* $k \geq 0$. *Then* $\max_{i,j}\{a_i/(a_i,a_j)\} \geq n$.

Following corollary 2, we have

COROLLARY 3 ([5]). *Let* A *be* p*-simple for a prime* $p \neq 2$ *and suppose that* A *contains a prime power* $a_k = p^d$ $(d \geq 0)$. *Then* $\max_{i,j}\{a_i/(a_i,a_j)\} \geq n$.

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LEMMA 4 ([4]). *If F is a finite collection of sets then the number of distinct differences of members of F is at least as large as the number of members of F.*

THEOREM 5. *Let $P(A) = \{p_1, p_2, ..., p_m\}$ and let $a_i = p_1^{i_1} p_2^{i_2} ... p_m^{i_m}$ for all a_i. If nonzero numbers of $\{1_j, 2_j, ..., n_j\}$ are equal for all p_j, then $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$.*

Proof. Let $F_i = \{p \in P(A) : p \mid a_i\}$, $i = 1, 2, ..., n$. Then $a_i = a_j$ if and only if $F_i = F_j$ for all i, j. Clearly $F_1, F_2, ..., F_n$ are n different sets. It follows from Lemma 4 that the number of different members of $\{F_i \setminus F_j : i, j = 1, 2, ..., n\}$ is at least as large as n. Since $F_i \setminus F_j = \{p \in P(A) : p \mid a_i/(a_i, a_j)\}$ and since $F_i \setminus F_j = F_h \setminus F_k$ if and only if $a_i/(a_i, a_j) = a_h/(a_h, a_k)$ for $1 \leq i, j, h, k \leq n$, it follows that $\{a_i/(a_i, a_j) : i, j = 1, 2, ..., n\}$ contains at least n different numbers. Thus, $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$.

COROLLARY 6 ([4]). *If the members of A are squarefree integers then $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$.*

THEOREM 7. *If $[a_1, a_2, ..., a_n] \nmid [1, 2, ..., n]$, then*

$$\max_{i,j}\{a_i/(a_i, a_j)\} > n$$

where $[a_1, a_2, ..., a_n] = lcm\{a_1, a_2, ..., a_n\}$.

Proof. Suppose $[a_1, a_2, ..., a_n] \nmid [1, 2, ..., n]$. Then there exists some a_k such that $a_k \nmid [1, 2, ..., n]$. Let $P(A) = \{p_1, p_2, ..., p_m\}$ and $a_k = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$. Then there exists $p_s^{k_s}$ such that $p_s^{k_s} \nmid [1, 2, ..., n]$. Hence $p_s^{k_s} > n$. Since $(a_1, a_2, ..., a_n) = 1$, we have $(p_s, a_t) = 1$ for some a_t, and hence $(p_s^{k_s}, a_t) = 1$. Thus $a_k/(a_k, a_t) \geq p_s^{k_s} > n$, and therefore, $\max_{i,j}\{a_i/(a_i, a_j)\} > n$.

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