Pusan Kyŏngnam Math. J. 10(1994), No 2, pp 287–293

ON ANALYSES OF NEAR-RING MORPHISMS

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1.Introduction

In this paper we will subject the category of near-rings to the same type of analysis that we applied to the categories of sets, semigroups and groups.

A near-ring is R a set together with two binary operations, addition and multiplication which satisfy: (1) R is a group under addition with identity denoted by 0 (not necessarily abelian), (2) R is a semigroup under multiplication, (3) For all elements a, b and c in R we have: $(a + b) = ac + bc$. A subnear-ring of R is a subset S of R which under addition is a subgroup of R and under multiplication is a subsemigroup of R. A lot of examples of near-rings are introduced in the book of G.Pilz [5]. Because a near-ring is completely determined by its underlying set as well as its structure as an additive group and multiplicative semigroup, we see that a morphism $f : R \longrightarrow S$ from the near-ring R to the near-ring S must be a map from the underlying set of R to that of S of which is compatible with both the additive and multiplicative structure of R and S.

2. Analyses of near-ring morphism

We now point out some demonstrable properties of surjective and injective morphisms of near-rings that are analogs of properties of maps of sets, semigroups. There are important because they are often useful in showing that analogs of results already obtained for maps of sets, semigroups and groups also hold for morphisms of near-rings.

LEMMA 2.1. *Suppose X, Y and Z are semigroups and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are maps of the underlying sets of the semigroups*

Received October 24,1994.

such that the map $gf : X \longrightarrow Z$ is a morphism of semigroups. Then

(1) *If $f : X \longrightarrow Y$ is a surjective morphism of semigroups, then $g : Y \longrightarrow Z$ is also a morphism of semigroups.*

(2) *If $g : Y \longrightarrow Z$ is an injective morphism of semigroups, then the map $f : X \longrightarrow Y$ is also a morphism of semigroups.*

Proof. (1) Suppose $gf : X \longrightarrow Z$ and $f : X \longrightarrow Y$ are morphism and surjective morphism of semigroups respectively. Let y_1, y_2 in Y. Since f is surjective, there is x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Using gf is morphism and f is morphism, we have :

$$\begin{aligned} g(y_1y_2) &= g(f(x_1)f(x_2)) = g(f(x_1x_2)) = gf(x_1x_2) \\ &= gf(x_1)gf(x_2) = g(f(x_1))g(f(x_2)) = g(y_1)g(y_2). \end{aligned}$$

Hence g is a morphism of semigroups.

(2) Suppose $gf : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ are morphism and injective morphism of semigroups respectively, and let x_1 and x_2 are in X. Then we have:

$$\begin{aligned} g(f(x_1)f(x_2)) &= g(f(x_1))g(f(x_2)) = gf(x_1)gf(x_2) \\ &= gf(x_1x_2) = g(f(x_1x_2)). \end{aligned}$$

Since g is injective, this follows that $f(x_1x_2) = f(x_1)f(x_2)$. Consequently, f is a morphism of semigroups.

As an application of how these observations can be used, we establish for semigroups the analog of following result which is obtained for sets.

LEMMA 2.2. *Suppose we are given a diagram of morphisms of semigroups,*

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & & \searrow g & \\ U & & & & V \\ & s \searrow & & \nearrow t & \\ & & Y & & \end{array}$$

satisfying :

(1) $gf = ts$;

(2) f *is a surjective morphism and* t *is an injective morphism. Then there is one and only one morphism of semigroups* $h : X \longrightarrow Y$ *which makes the diagram commutative, that is, such that* $hf = s$ *and* $th = g$.

Proof. Considering the diagram

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & & \searrow g & \\ U & & & & V \\ & s \searrow & & \nearrow t & \\ & & Y & & \end{array}$$

simply as a daigram of maps of sets, the hypothesis that f is a surjective map and t is an injective map such that $gf = ts$ implies that there is a unique map of sets $h : X \longrightarrow Y$ such that $hf = s$ and $th = g$ [Hint: For each x in X, Choose u in U such that $f(u) = x$, show that the element $s(u)$ in Y is independent of the choice of u and define $h(x)$ to be $s(u)$].

Thus we have the commutative diagram;

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & & \searrow g & \\ U & & \downarrow h & & V \\ & s \searrow & & \nearrow t & \\ & & Y & & \end{array}$$

Since $thf = ts$ it follows thf is a morphism of semigroups. Because t is an injective morphism and the composition $t(hf)$ is also a morphism, it follows from our previous Lemma 2.1 that the map hf is a morphism of semigroups. Hence, by the some Lemma, the fact that f is a surjective morphism and hf is a morphism of semigroups implies that the map h is really a morphism of semigroups. Now it is not difficult to check that the morphism $h : X \longrightarrow Y$ has our desired properties and is the only morphism from X to Y having these properties.

Let R and S be near-rings. Then we have the following basic properties : If R is a subring of S, then the inclusion map of sets $R \longrightarrow S$

is an injective morphism of near-rings called the inclusion morphism and written as inc : $R \longrightarrow S$.

Now suppose $f : R \longrightarrow S$ is an arbitrary morphism of near-rings. Then Imf is a subnear-ring called of S called the image of f, and the map $\overline{f} : R \longrightarrow Imf$ is a surjective morphism of near-rings, so that the morphism $f : R \longrightarrow S$ is the composition of the morphisms of near-rings.

$$R \xrightarrow{\overline{f}} Imf \xrightarrow{\text{inc}} S$$

This factorization of f is called the image analysis of f. More generally, any factorization

$$R \xrightarrow{g} R' \xrightarrow{h} S$$

of f with g a surjective morphism of near-rings and h an injective morphism of near-rings is called an analysis of f.

PROPOSITION 2.3. *Suppose R, S and T are near-rings and $f : R \longrightarrow S$ and $g : S \longrightarrow T$ are maps of underlying sets of the near-rings such that the map $gf : R \longrightarrow T$ is a morphism of near-rings. Then*

(1) *If $f : R \longrightarrow S$ is a surjective morphism of near-rings, then $g : S \longrightarrow T$ is also a morphism of near-rings.*
(2) *If $g : S \longrightarrow T$ is an injective morphism of near-rings, then $f : R \longrightarrow S$ is also a morphism of near-rings.*

Proof. (1) Because the composition $gf : R \longrightarrow T$ is a morphism of near-rings, it is certainly a morphism of the additive group of R to the additive group of T. Similarly, $f : R \longrightarrow S$ is a surjective morphism of the additive group of R to that of S. Hence, by the Lemma 2.1, concerning semigroups, we know that the map $g : S \longrightarrow T$ is also a morphism of the additive group of S to the additive group of T. A similar argument also shows that $g : S \longrightarrow T$ is a morphism of the multiplicative semigroup of S to that of T. Therefore the map $g : S \longrightarrow T$ is a morphism of near-rings because it is both a morphism of the additive groups of S and T, and a morphism of the multiplicative semigroups of S and T.

(2) This can be established in a manner similar to part (1) and is left to the readers.

As for semigroups in Lemma 2.2, We have, as a direct consequence of this result, the following.

PROPOSITION 2.4. *Suppose we are given a commutative diagram of morphisms of near-rings,*

$$\begin{array}{ccccc} & & S & & \\ & f \nearrow & & \searrow g & \\ R & & & & T \\ & f' \searrow & & \nearrow g' & \\ & & S' & & \end{array}$$

satisfying :

(1) f is a surjective morphism;
(2) g' is an injective morphism.

Then there is one ans only one morphism of near-rings $h : S \longrightarrow S'$ such that the diagram :

$$\begin{array}{ccccc} & & S & & \\ & f \nearrow & & \searrow g & \\ R & & \downarrow h & & T \\ & f' \searrow & & \nearrow g' & \\ & & S' & & \end{array}$$

commutes.

By way of application of this result we show that any two analyses of a morphism of near-rings are essentially the same. This case follows from the proposition 2.3 and proposition 2.4.

PROPOSITION 2.5. *If $R \xrightarrow{g} S \xrightarrow{h} T$ and $R \xrightarrow{g'} S' \xrightarrow{h'} T$ are analysys of the same morphism of near-rings $f : R \longrightarrow T$ then there is a unique morphism of near-rings $j : S \longrightarrow S'$, such that the diagram*

$$\begin{array}{ccccc} & & S & & \\ & g \nearrow & & \searrow h & \\ R & & \downarrow j & & T \\ & g' \searrow & & \nearrow h' & \\ & & S' & & \end{array}$$

commutes. This uniquely determined morphism $j : S \longrightarrow S'$ is an isomorphism of near-rings.

Proof. Since $R \xrightarrow{g} S \xrightarrow{h} T$ and $R \xrightarrow{g'} S' \xrightarrow{h'} T$ are analyses of the same near-ring morphism $f : R \longrightarrow T$, it following that is a commutative diagram satisfying (1) g and g' are surjective morphisms and (2) h and h' are injective morphisms. Hence, by the proposition 2.4, there are unique near-ring morphisms $j : S \longrightarrow S'$ and $k : S' \longrightarrow S$ such that the diagrams

$$\begin{array}{ccccc} & & S & & \\ & g \nearrow & & \searrow h & \\ R & & \downarrow j & & T \\ & g' \searrow & & \nearrow h' & \\ & & S' & & \end{array}$$

and

$$\begin{array}{ccccc} & & S' & & \\ & g \nearrow & & \searrow h & \\ R & & \downarrow k & & T \\ & g' \searrow & & \nearrow h' & \\ & & S & & \end{array}$$

commute. If we show that this uniquely determined morphism $j : S \longrightarrow S'$is an isomorphism, we will have established our desired result. It follows from the commutativity of the above diagrams, that the diagram

$$\begin{array}{ccccc} & & S & & \\ & g \nearrow & & \searrow h & \\ R & & \downarrow kj & & T \\ & g \searrow & & \nearrow h & \\ & & S & & \end{array}$$

commutes. But by the previous proposition 2.4, we know that there is only one morphism $S \longrightarrow S$ which makes the diagram

$$\begin{array}{ccccc} & & S & & \\ & g \nearrow & & \searrow h & \\ R & & & & T \\ & g \searrow & & \nearrow h & \\ & & S & & \end{array}$$

commute because g is a surjective morphism and h is an injective morphism. Therefore, the fact that both the identity morphism $1_s : S \longrightarrow$

S and $kj : S \longrightarrow S$ have property implies that $kj = 1_s$. A similar argument shows that $jk = 1_{s'}$. Therefore, we have established that $j : S \longrightarrow S'$ is an isomorphism of near-ring, which completes the proof of the proposition.

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