Pusan Kyŏngnam Math. J. 10(1994), No. 2, pp. 279–285

A NOTE ON HILBERT ALGEBRAS

Y. B. Jun, J. W. Nam and S. M. Hong

The concept of Hilbert algebras was introduced by Diego [5]. Busneag [2 - 4] developed the theory of Hilbert algebras and deductive systems. The aims of this paper are

1. To prove that if $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are distinct elements of a Hilbert algebra, then there exists x_i in Hilbert algebra such that $x_j \to x_i \neq 1$ for all $j \neq i$;

2. To show that any bounded Hilbert algebra of order $n+1$ must contain a Hilbert subalgebra of order $n \geq 2$;

3. To verify that the set of all involutions of a bounded Hilbert algebra is a Hilbert subalgebra; and

4. To define the concept of stabilizers and prove some propositions.

Definition 1 ([3, 5]). A *Hilbert algebra* is a triple $(H, \to, 1)$, where H is a nonempty set, $\to$ is a binary operation on H, $1 \in H$ is an element (called a unit element) such that the following three axioms are satisfied for every $x, y, z \in H$:

(a_1) $x \to (y \to x) = 1$,

(a_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(a_3) If $x \to y = y \to x = 1$ then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H, which will be called the *natural ordering* on H; with respect to this order relation, 1 is the largest element of H.

A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0 relative to natural ordering.

A subset S of a bounded Hilbert algebra H is called a *Hilbert subalgebra* of H if $0 \in S$ and $x, y \in S \Rightarrow x \to y \in S$.

Examples ([3]). 1. If $(H, \leq)$ is a poset, then $(H, \to, 1)$ is a Hilbert algebra, where 1 is the largest element of H and

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$$

Received October 5,1994.

for $x, y \in H$.

2. If $(H, \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice, then $(H, \rightarrow, 1)$ is a bounded Hilbert algebra, where $\rightarrow$ is defined by $x \rightarrow y = (\neg x) \vee y$ for $x, y \in H$.

PROPOSITION 2 ([3, 5]). *If H is a Hilbert algebra and $x, y, z \in H$, then the following hold:*

(b_1) $x \leq y \rightarrow x$.
(b_2) $x \rightarrow 1 = 1$.
(b_3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.
(b_4) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$.
(b_5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
(b_6) $x \leq (x \rightarrow y) \rightarrow y$, $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.
(b_7) $1 \rightarrow x = x$.
(b_8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
(b_9) *If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.*

PROPOSITION 3. *In a Hilbert algebra H, we have the following properties:*

(c_1) *If $x \neq y$, then $y \rightarrow x \neq 1$ whenever $x \rightarrow y = 1$.*
(c_2) *If $x \leq y$ and $y \leq z$ then $x \leq z$.*

Proof. (c_1) Assume that $x \neq y$ and $x \rightarrow y = 1$. If $y \rightarrow x = 1$ then by (a_3) we have $x = y$. This is impossible.

(c_2) Suppose $x \leq y$ and $y \leq z$. Applying (b_9), we obtain $1 = x \rightarrow y \leq x \rightarrow z$, and so $x \rightarrow z = 1$ or $x \leq z$. The proof is complete.

For an n sequence $x_1, x_2, ..., x_n$ of a Hilbert algebra H, consider the $(n-1) \times n$ matrix

$$A = \begin{pmatrix} x_2 \rightarrow x_1 & x_1 \rightarrow x_2 & \dots & x_1 \rightarrow x_n \\ x_3 \rightarrow x_1 & x_3 \rightarrow x_2 & \dots & x_2 \rightarrow x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n \rightarrow x_1 & x_n \rightarrow x_2 & \dots & x_{n-1} \rightarrow x_n \end{pmatrix}.$$

For convenience in this paper, we call A the *H-matrix* relative to the n sequence $x_1, x_2, ..., x_n$.

THEOREM 4. *Let $x_1, x_2, ..., x_n$ $(n \geq 2)$ be a n sequence of a Hilbert algebra H. If $x_i \neq x_j$ whenever $i \neq j$ $(1 \leq i, j \leq n)$ then there exists a column in the H-matrix A which consists of nonunit elements.*

Proof. The proof will be by induction on n. If $n = 2$ then the H-matrix is

$$A = (x_2 \to x_1, x_1 \to x_2).$$

Assume that $x_2 \to x_1 = x_1 \to x_2 = 1$. Then, by (a_3), we have $x_1 = x_2$, which is a contradiction. Hence the assertion holds for $n = 2$. Suppose that the theorem is true for $n = k$. For a $k+1$ sequence $x_1, x_2, ..., x_k, x_{k+1}$ of H, let $x_i \neq x_j$ whenever $i \neq j$ $(1 \leq i, j \leq k+1)$ and let

$$A_{k+1} = \begin{pmatrix} x_2 \to x_1 & x_1 \to x_2 & \dots & x_1 \to x_k & x_1 \to x_{k+1} \\ x_3 \to x_1 & x_3 \to x_2 & \dots & x_2 \to x_k & x_2 \to x_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_k \to x_1 & x_k \to x_2 & \dots & x_{k-1} \to x_k & x_{k-1} \to x_{k+1} \\ x_{k+1} \to x_1 & x_{k+1} \to x_2 & \dots & x_{k+1} \to x_k & x_k \to x_{k+1} \end{pmatrix}.$$

be its H-matrix. Denote

$$A_k = \begin{pmatrix} x_2 \to x_1 & x_1 \to x_2 & \dots & x_1 \to x_k \\ x_3 \to x_1 & x_3 \to x_2 & \dots & x_2 \to x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k \to x_1 & x_k \to x_2 & \dots & x_{k-1} \to x_k \end{pmatrix}.$$

Then A_k is the H-matrix relative to the k sequence $x_1, x_2, ..., x_k$. By inductive hypothesis, there exists a column in A_k, which consists of nonunit elements. Without loss of generality we may assume that

$$\begin{cases} x_2 \to x_1 \neq 1, \\ x_3 \to x_1 \neq 1, \\ \quad \vdots \\ x_k \to x_1 \neq 1. \end{cases}$$

Now it suffices to discuss the following cases:

If $x_{k+1} \to x_1 \neq 1$ then each element in the first column of A_{k+1} is not equal to 1, and we obtain the results.

Assume $x_{k+1} \to x_1 = 1$. Since $x_1 \neq x_{k+1}$, it follows from (c_1) that $x_1 \to x_{k+1} \neq 1$. We then claim that $x_i \to x_{k+1} \neq 1$ for all $2 \leq i \leq k$. In fact, if not, then there exists i_0 $(2 \leq i_0 \leq k)$ such that

$x_{i_0} \to x_{k+1} = 1$. Using (c_2), we get $x_{i_0} \to x_1 = 1$. This is impossible, and so $x_i \to x_{k+1} \neq 1$ for all $1 \leq i \leq k$. Hence the assertion holds for $n = k+1$. This finishes the proof of the theorem.

For a set S denote the cardinal of S by $|S|$. For a Hilbert algebra $(H; \to, 1)$, $|H|$ is called to be the *order* of this algebra. If $|H| < \infty$, then $(H; \to, 1)$ is called to be of *finite order*; if $|H| = n$, then it is said to be of *order n*; if $|H| = \infty$, then it is said to be of *infinite order*.

THEOREM 5. *Let H be a bounded Hilbert algebra. If S is a Hilbert subalgebra of H, then $|S| \geq 2$.*

Proof. We note that $0 \in S$, so that $0 \to 0 = 1 \in S$. Hence $|S| \geq 2$.

THEOREM 6. *Any bounded Hilbert algebra of order $n+1$ must contain a Hilbert subalgebra of order $n \geq 2$.*

Proof. Let $H = \{0, x_1, x_2, ..., x_{n-1}, 1\}$ be a bounded Hilbert algebra of order $n+1$, in which $x_i \neq x_j$ whenever $i \neq j$ and let

$$B = \begin{pmatrix} x_2 \to x_1 & x_1 \to x_2 & \dots & x_1 \to x_{n-1} \\ x_3 \to x_1 & x_3 \to x_2 & \dots & x_2 \to x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} \to x_1 & x_{n-1} \to x_2 & \dots & x_{n-2} \to x_{n-1} \end{pmatrix}$$

be the H-matrix relative to the $n-1$ sequence $x_1, x_2, ..., x_{n-1}$. By Theorem 4, there exists a column in B which consists of nonunit elements. Without loss of generality, we can suppose it is the last column of B, that is,

$$x_i \to x_{n-1} \neq 1, \text{ for all } i = 1, 2, ..., n-2.$$

We now show that $H' = \{0, x_1, ..., x_{n-2}, 1\}$ is a Hilbert subalgebra of H. If H' is not a Hilbert subalgebra of H, then there exist distinct subscripts i and j $(1 \leq i, j \leq n-2)$ such that $x_i \to x_j = x_{n-1}$. It follows from (a_1) that

$$x_j \to x_{n-1} = x_j \to (x_i \to x_j) = 1,$$

which contradicts to $x_i \to x_{n-1} \neq 1$ for all $i = 1, 2, ..., n-2$. This completes the proof.

THEOREM 7. *Let H be a bounded Hilbert algebra of order $n(\geq 2)$ and let $N(k)$ denote the number of Hilbert subalgebras of order k in H. Then $1 \leq N(k) \leq \binom{n-2}{k-2}$ for $k = 2, ..., n$.*

Proof. We note that any Hilbert subalgebra of order $k(2 \leq k \leq n)$ consists of $0, 1$ and $k-2$ elements $(\neq 0, 1)$. Since there are $n-2$ elements $(\neq 0, 1)$ in H, therefore $N(k) \leq \binom{n-2}{k-2}$. Using Theorem 6, We have $1 \leq N(k)$ for all $k(2 \leq k \leq n)$. The proof is complete.

If H is a bounded Hilbert algebra and $x \in H$, we denote by $x^* = x \to 0$.

PROPOSITION 8 ([3]). *Let H be a bounded Hilbert algebra and $x, y \in H$. Then*

(b_{10}) $0^* = 1$, $1^* = 0$.
(b_{11}) $x \to y^* = y \to x^*$.
(b_{12}) $x \to x^* = x^*$, $x^* \to x = x^{**}$.
(b_{13}) $x \to y \leq y^* \to x^*$.
(b_{14}) $x \leq y$ *implies* $y^* \leq x^*$.
(b_{15}) $(x \to y)^{**} = x^{**} \to y^{**}$.

PROPOSITION 9. *In a bounded Hilbert algebra H, the following properties hold:*

(c_3) $x \leq x^{**}$.
(c_4) $x^{***} = x^*$.

Proof. Using (b_6), we have $x \leq (x \to 0) \to 0 = x^* \to 0 = x^{**}$ and $x^* = x \to 0 = ((x \to 0) \to 0) \to 0 = (x^* \to 0) \to 0 = x^{**} \to 0 = x^{***}$. This completes the proof.

DEFINITION 10. Let H be a bounded Hilbert algebra. If an element x of H satisfies $x^{**} = x$, then x is called an *involution*.

Denote by $\mathcal{I}(H)$ the set of all involutions of H. Since $1^{**} = 1$, the element 1 is contained in $\mathcal{I}(H)$. Hence $\mathcal{I}(H)$ is not empty.

THEOREM 11. *For any bounded Hilbert algebra H, $\mathcal{I}(H)$ is a Hilbert subalgebra of H.*

Proof. Let $x, y \in \mathcal{I}(H)$. Using (b_{15}); then we have

$$\begin{aligned}(x \to y)^{**} \to (x \to y) &= (x^{**} \to y^{**}) \to (x \to y)\\ &= (x \to y) \to (x \to y) = 1,\end{aligned}$$

that is, $(x \to y)^{**} \leq x \to y$. It follows from (c_3) and (a_3) that $(x \to y)^{**} = x \to y$, which means that $x \to y \in \mathcal{I}(H)$. Since $0^{**} = 0$, we have $0 \in \mathcal{I}(H)$. Hence $\mathcal{I}(H)$ is a Hilbert subalgebra of H.

COROLLARY 12 ([2]). *If H is a bounded Hilbert algebra, then the following assertions are equivalent:*

(c_5) *H is a Boolean lattice relative to natural ordering.*

(c_6) *Every element of H is involution.*

DEFINITION 13. Let H be a Hilbert algebra. For $a \in H$, the set $H_a := \{x \in H | x \to a = a\}$ is called the *stabilizer* of a in H.

The following theorem is obvious.

THEOREM 14. *For a Hilbert algebra H and $a \in H$, we have*

$$H_a = H \text{ if and only if } a = 1 \text{ if and only if } a \in H_a.$$

DEFINITION 15 ([5]). If H is a Hilbert algebra, a subset D of H is called a *deductive system* of H if it satisfies:

(a_4) $1 \in D$,

(a_5) $x \in D$ and $x \to y \in D$ imply $y \in D$.

THEOREM 16. *For a Hilbert algebra H and $a \in H$, the stabilizer H_a of a is a deductive system of H.*

Proof. Since $1 \to a = a$, we have $1 \in H_a$. Let $x \in H_a$ and $x \to y \in H_a$. Then $x \to a = a$ and $(x \to y) \to a = a$. From (b_1), we get $a \leq y \to a$. Now

$$\begin{aligned}
(y \to a) \to a &= (y \to (x \to a)) \to a \\
&= (x \to (y \to a)) \to a \qquad [\text{by } (b_5)] \\
&= ((x \to y) \to (x \to a)) \to a \qquad [\text{by } (b_3)] \\
&= (x \to ((x \to y) \to a)) \to a \qquad [\text{by } (b_5)] \\
&= (x \to a) \to a \\
&= a \to a \\
&= 1,
\end{aligned}$$

and so $y \to a \leq a$. Hence $y \to a = a$ or $y \in H_a$. This completes the proof.

THEOREM 17. *In a Hilbert algebra of order n, any deductive system of order $n-1$ is the stabilizer of some element.*

Proof. Let H be a Hilbert algebra of order n and let D be a deductive system of order $n-1$. Assume that $a \notin D$. Then for any $x \in D$, $x \to a \notin D$. Hence $x \to a = a$ or $x \in H_a$. Since $a \neq 1$, we have $a \notin H_a$. Therefore $D = H_a$, completing the proof.

THEOREM 18. *Let H be a Hilbert algebra of order n and let D be a proper deductive system of H. Then there exists an element $a(\neq 1)$ in H such that $D \subseteq H_a$.*

Proof. Since $H \setminus D = \emptyset$ and $|H \setminus D| < \infty$, there exists a maximal element, say a, in $H \setminus D$. It is obvious that $a \neq 1$. For any $x \in D$, $x \to a \notin D$ or $x \to a \in H \setminus D$. Since a is a maximal element in $H \setminus D$, by using (b_1) we have $x \to a = a$ or $x \in H_a$. This means that $D \subseteq H_a$ which completes the proof.

References

1. R. Balbes and P. Dwinger, *Distributive lattice*, University of Missouri Press (1974).
2. D. Busneag, *A note on deductive systems of a Hilbert algebra*, Kobe J. Math **2** (1985), 29 - 35
3. D. Busneag, *Hilbert algebras of fractions and maximal Hilbert algebras of quotients*, Kobe J. Math. **5** (1988), 161 - 172.
4. D. Busneag, *Hertz algebras of fractions and maximal Hertz algebras of quotients*, Math Japon. **39** (1993), 461 - 469.
5. A. Diego, *Sur les algébras de Hilbert*, Ed. Hermann, Colléction de Logique Math Serie A **21** (1966).
6. G. Gratzer, *Lattice theory*, W H. Freeman and Company, San Francisco (1979).
7. J. Hashimoto, *Ideal theory for lattices*, Math. Japon **2** (1952), 149 - 186.
8. Y. B. Jun, *Deductive systems of Hilbert algebras*, to appear in Math. Japon
9. J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Seoul, Korea (1994).

Department of Mathematics (Education)
Gyeongsang National University
Chinju 660-701, Korea