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ON THE UNIQUENESS OF SEQUENTIAL LIMITS IN SEQUENTIAL CONVERGENCE SPACES

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1. Introduction

It is well known that every compact subset of a Hausdorff space is closed. In [4], A. J. Insel obtained the following result:

THEOREM [4]. *Let X be a first-countable space. X is Hausdorff if and only if every compact subset of X is closed.*

In this note, we shall introduce sequential convergence spaces and study the relationship between Hausdorffness and uniqueness of sequential limits in a sequential convergence space. We shall obtain above A. J. Insel's reault for the case in which we replace a first-countable space (Hausdorffness) by a sequential convergence space (resp. uniqueness of sequential limits). Moreover, we shall prove that a sequential convergence space X has unique sequential limits if and only if every sequentially compact subset of X is closed.

2. Sequential Convergence Spaces

In this section, we introduce sequential convergence spaces. Let X be any non-empty set and let $S(X)$ be the set of all sequences in X. A non-empty subfamily L of $S(X) \times X$ is called *a sequential convergence structure on X* if it satisfies the following properties:

$(SC1)$ For each $x \in X$, $((x), x) \in L$, where (x) is the constant sequence whose the k-th term is x for all indices k.

$(SC2)$ If $((x_n), x) \in L$, then $((x_{\phi(n)}), x) \in L$ for each subsequence $(x_{\phi(n)})$ of (x_n).

$(SC3)$ Let $x \in X$ and $A \subset X$. If $((y_n), x) \in L$ for some (y_n) in $\{y \in X | ((z_n), y) \in L$ for some (z_n) in $A\}$, then $((x_n), x) \in L$ for some (x_n) in A.

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THEOREM 2.1 [3]. *Let L be a sequential convergence structure on X. Define a function $C_L : P(X) \to P(X)$ by*

$$C_L(A) = \{x \in X | ((x_n), x) \in L \text{ for some } (x_n) \text{ in } A\}$$

for each subset A of X, where $P(X)$ is the power set of X. Then C_L is a Kuratowski closure operator on X and hence (X, C_L) is a topological space.

Hereafter, we use the notation (X, L) for (X, C_L) and we call the pair (X, L) **a sequential convergence space** *[3].*

In a topological space X, the following property is called *Fréchet-Urysohn* [1, 5, 7];

> Let $A \subset X$ and $x \in X$. $x \in Cl(A)$ if and only if (x_n) converges to x in X for some sequence (x_n) in A, where $Cl(A)$ is the closure of A.

REMARK. *(1) It is well known that every first-countable space, and hence each metric space and each discrete space, has the Fréchet-Urysohn property.*

(2) A topological space is called a Fréchet space [7] (or a Fréchet-Urysohn space [5]) if it has the Fréchet-Urysohn property.

(3) Every sequential convergence space has the Fréchet-Urysohn property by Theorem 2.1.

THEOREM 2.2. *Let (X, τ) be a topological space. If X has the Fréchet-Urysohn property, then L_τ is a sequential convergence structure on X, where*

$$L_\tau = \{((x_n), x) \in S(X) \times X | (x_n) \text{ converges to } x \text{ in } X\}.$$

Proof. (SC1) and (SC2) are obvious. (SC3): Let $A \subset X$ and $x \in X$. Assume that $((y_n), x) \in L_\tau$ for some (y_n) in

$$\{y \in X | ((z_n), y) \in L_\tau \text{ for some } (z_n) \text{ in } A\}$$

Then, since X has the Fréchet-Urysohn property,

$$\{y \in X | ((z_n), y) \in L_\tau \text{ for some } (z_n) \text{ in } A\} = C_{L_\tau}(A),$$

where $C_{L_\tau}(A)$ is the Kuratouski closure operator on X defined by L_τ as Theorem 2.1, and $x \in C_{L_\tau}(C_{L_\tau}(A)) = C_{L_\tau}(A)$. Thus, $((x_n), x) \in L_\tau$ for some (x_n) in A.

It is easy to verify that $\{U \subset X | C_{L_\tau}(X - U) = X - U\} = \tau$, and hence we obtain that two spaces (X, τ) and (X, L_τ) have exactly same topology. Consequently, we have that every topological space having the Fréchet-Urysohn property is a sequential convergence space.

THEOREM 2.3 [3]. *Let (X, L) be a sequential convergence space and $x \in A \subset X$. Then, A is a neighborhood of x in (X, L) if and only if for each $((x_n), x) \in L$, (x_n) is eventually in A.*

COROLLARY 2.4. *Every first-countable space , and hence each metric space and each discrete space, is a sequential convergence space.*

Proof. Let (X, τ) be a first- countable space and let

$$L_\tau = \{((x_n), x) \in S(X) \times X | (x_n) \text{ converges to } x \text{ in } X\}.$$

Then, it is easy to prove that L_τ is a sequential convergence structure on X and

$$\tau = \{U \subset X | C_{L_\tau}(X - C) = X - U\}.$$

Thus (X, τ) is a sequential convergence space.

Next, we show that the converse of above corollary is not true, in general.

EXAMPLE 2.5. *Let X be a uncountable discrete space. Then, the one-point compactification X^*, let $X^* = X \cup \{\infty\}$, of X is a sequential convergence space, but not first-countable.*

Proof. First, we show that X^* is a sequential convergence space. By Theorem 2.2, it is sufficient to prove that X^* has the Fréchet-Urysohn property. Let $A \subset X^*$ and $z \in Cl_{X^*}(A)$. If $Cl_{X^*}(A) = A$, then there exists the constant sequence (z) in A with (z) converges to z in X^* and hence it holds. We divide this proof into two cases:

Case 1: If $\infty \in A$, then it is clear that $Cl_{X^*}(A) = A$. For, if $z \in Cl_{X^*}(A) - A$, then $z \in X$ and hence $\{z\}$ is open in X(and open in X^*), impossible.

Case 2: If $A \subset X$, then either $A = Cl_{X^*}(A)$ or $A \cup \{\infty\} = Cl_{X^*}(A)$. Hence we must prove this for the case $A \cup \{\infty\} = Cl_{X^*}(A)$, i.e., $z = \infty$.

Since $\infty \in Cl_{X^*}(A)$, $U \cap A \neq \emptyset$ for each open set U containing ∞ in X^*. Since X is discrete, every open set containing ∞ in X^* is the form $X^* - F$, where F is a finite subset of X. It follows that A is infinite and for every infinitely countable subset $\{x_n | n \in N\}$ in A, the sequence (x_n) in A converges to ∞ in X^*. Thus, X^* has the Fréchet-Urysohn property. Next we show that X^* is not first-countable. Since X is an uncountable discrete space, the local base $\mathcal{B}_\infty$ of ∞ in X^* is a superset of the uncountable set $\{X^* - \{x\} | x \in X\}$ and hence X^* is not first-countable. Therefore X^* is a sequential convergence space, but not first-countable.

It is easy to verify that for a first-countable space X, X is Hausdorff if and only if X has unique sequential limits, i.e., every convergence sequence in X has only one limit point in X. It is also clear that if a topological space X is Hausdorff, then X has unique sequential limits, but the converse is not true in general. In [1], an example of a non first-countable Fréchet space (and hence a sequential convergence space), not Hausdorff, but with unique sequential limits was given as follows: Let $X = (N \times N) \cup \{p, q\}$ with $p \neq q$ and $\{p, q\} \cap (N \times N) = \emptyset$, where N is the set of all natural numbers. Each $(i, j) \in N \times N$ will be discrete. Basic open neighborhoods of p will be of the form $\{p\} \cup \bigcup_{i \geqslant k} \{(i, j) \mid j \in N\}$ for each $k \in N$, and those of q of the form $\{q\} \cup \bigcup_i \{(i, j) \mid j \geqslant j_i \in N\}$.

3. Main Results

We now shall show A. J. Insel's result for the case in which we replace a first-countable space (Hausdorffness) by a sequential convergence space (resp. uniqueness of sequential limits). The proof of this theorem is very similar to Theorem 3.3 below and hence we omit.

THEOREM 3.1. *Let (X, L) be a sequential convergence space. Then, X has unique sequential limits if and only if every compact subset of X is closed.*

Combining Corollary 2.4 and Theorem 3.1, we have A. J. Insel's result.

COROLLARY 3.2 [4]. *Let X be any first-countable space. Then, X is Hausdorff if and only if every compact subset of X is closed.*

Finally. we shall obtain Theorem 3.1 for the case in which we replace compactness by sequential compactness.

REMARK. *(1) There are compact Hausdorff spaces that are not sequentially compact (See [2, p.313]).*

(2) There are sequentially compact Hausdorff, first-countable spaces that are not compact (See [6, p.163]).

THEOREM 3.3. *Let (X, L) be a sequential convergence space. Then, X has unique sequential limits if and only if every sequentially compact subset of X is closed.*

Before we prove this theorem, we show the following lemma.

LEMMA 3.4. *Let X be any topological space and let (x_n) be a sequence in X with $x_n \to x \in X$. Then, let $A = \{x_n | n \in N\} \cup \{x\}$, A is a sequentially compact subset of X.*

Proof. Let (y_n) be a sequence in A. If the range $\{y_n | n \in N\}$ of (y_n) is finite, it is obvious that (y_n) has a convergent subsequence. Assume that the range $\{y_n | n \in N\}$ of (y_n) is infinite. If there exists $z \in \{y_n | n \in N\}$ such that $\{n \in N | z = y_n\}$ is infinite, then it is clear that the constant sequence (z) is a convergence subsequence of (y_n). If (y_n) does not have any constant subsequence, we can construct a subsequence $(y_{\phi(n)})$ of (y_n) using the well-orderedness of the natural number set N as follows : Put α_1=the first element of $\{n \in N | x_n \in \{y_n | n \in N\}\}$ and $\phi(1) = max\{n \in N | x_{\alpha_1} = y_n\}$. Put α_2=the first element of $\{n \in N | x_n \in \{y_n | n > \phi(1)\}\}$ and $\phi(2) = max\{n \in N | x_{\alpha_2} = y_n,\ n > \phi(1)\}$. By induction, for each $p \in N(p \geqq)$, we can take

$$\alpha_p = \text{ the first element of } \{n \in N | x_n \in \{y_n | n > \phi(p-1)\}\}$$

and

$$\phi(p) = \max\{n \in N | x_{\alpha_p} = y_n,\ n > \phi(p-1)\}.$$

Then, clearly, this sequence $(y_{\phi(n)})$ is a subsequence of (y_n) and also a subsequence of (x_n), and thus $y_{\phi(n)} \to x$ because $x_n \to x$.

Proof of Theorem 3.3. Suppose that there is a sequence (x_n) in X such that $x_n \to p$ and $x_n \to q$ with $p \neq q$. We divide this proof into two cases.

Case 1. $\{n \in N | x_n = p\}$ is infinite (or $\{n \in N | x_n = q\}$ is infinite).

Then, clearly, the constant sequence (p) is a subsequence of (x_n). Since X is a sequential convergence space, by (SC 2), (p) also converges to p and q. It is clear that $\{p\}$ is sequentially compact and hence, by hypothesis, $\{p\}$ is closed and so $\{p\} = C_L(\{p\})$, which is a contradiction to the fact that the constant sequence (p) converges to p and q with $p \neq q$.

Case 2. $\{n \in N | x_n = p \text{ or } x_n = q\}$ is finite.

Then, there exists a subsequence $(x_{\phi(n)})$ of (x_n) such that $x_{\phi(n)} \neq p$, $x_{\phi(n)} \neq q$ for all $n \in N$. Since X is a sequential convergence space and (x_n) converges to p and q , by (SC2), $(x_{\phi(n)})$ also converges to p and q. Let $A = \{x_{\phi(n)} | n \in N\}$. Then, by above Lemma 3.4, $A \cup \{p\}$ is sequentially compact. By hypothesis, $A \cup \{p\}$ is closed, and so $A \cup \{p\} = C_L(A \cup \{p\})$. It follows that $q \notin C_L(A \cup \{p\})$ and hence there does not exist a sequence (y_n) in $A \cup \{p\}$ such that $y_n \to q$, which is a contradiction to the fact that the sequence $(x_{\phi(n)})$ in A converges to p and q with $p \neq q$. Therefore, we have that every convergent sequence in X has only one limit, and so X is Hausdorff.

Conversely, assume that X has unique sequential limits and let A be a sequentially compact subset of X. It is enough to show that $C_L(A) \subset A$. Let $x \in C_L(A)$. Then, since X is a sequential convergence space, X has the Fréchet-Urysohn property and hence there exists a sequence (x_n) in A such that $x_n \to x$. Since A is sequentially compact; every sequence in A has a convergent subsequence which converges to a point in A, it follows that the limit of any convergent sequence in A is in A. Thus we have that $x \in A$, and therefore A is closed.

Combining Corollary 2.4 and Theorem 3.3, we have the following.

COROLLARY 3.5. *Let X be a first-countable space. Then, X is Hausdorff if and only if every sequentially compact subset of X is closed.*

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