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SOME NOTES ON POSITIVE MAP

AN-HYUN KIM

1. Introduction

Throughout this note, C^*-algebras possess a unit and are written in type U, B. $B(H)$ denotes the algebra of all bounded operators on the Hilbert space H. M_n denotes the algebra of all $n \times n$ complex matrices, $M_n(U)$ the algebra of $n \times n$ matrices over U.

For an operator $T \in B(H)$, T said to be *positive*(in notation $T \geq 0$) if $T = T^*$ and the spectrum of T is included in $[0, \infty)$. In the matrix case, a squre matrix $A \in M_n$ is *positive* if $A = A^*$(the transpose of the complex conjugate) and all its eigenvalues are nonnegative. This note is an investigation of positive maps, and main results are Theorem 2.6, Corollary 2.7, Corollary 3.5.

In general, every unital positive map is contractive([1],[2],[5],[10]) and it has been generalized by M.D. Choi([5],[10], Theorem 2.5).

In section 2, we try to introduce the Choi's generalization, and exhibit some conditions for positive matrix and positive map(Lemma 2.2, Theorem 2.6). In section 3, we consider some particular positive maps and give a case that Theorem 2.5 does not hold for hyponormal element(Corollary 3.5).

2. Some conditions for positive matrix

LEMMA 2.1([5]). *Let $R, S, T \in B(H)$ with T being positive and invertible. Then $\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0$ if and only if $R \geq S^* T^{-1} S$.*

LEMMA 2.2. *Let $R, S, T \in B(H)$ with T and R positive. Then $\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0$ if and only if $| < Sy, x > |^2 \leq < Tx, x > \cdot < Ry, y >$ for all x, y in H.*

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Proof. ($\Rightarrow$) For any $\lambda \in C$,

$$\left\langle \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \begin{pmatrix} \lambda x \\ y \end{pmatrix}, \begin{pmatrix} \lambda x \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \lambda Tx + Sy \\ \lambda S^* x + Ry \end{pmatrix}, \begin{pmatrix} \lambda x \\ y \end{pmatrix} \right\rangle$$
$$= |\lambda|^2 < Tx, x > + \overline{\lambda} < Sy, x > + \lambda < x, Sy > + < Ry, y >$$
$$\geq 0.$$

If $< Sy, x >= re^{i\theta}$, then letting $\lambda = te^{i\theta} (t \in R)$,

$$t^2 < Tx, x > + te^{-i\theta} re^{i\theta} + te^{i\theta} re^{-i\theta} + < Ry, y > \geq 0.$$

$$t^2 < Tx, x > +2tr + < Ry, y > \geq 0.$$

Thus $R^2 - < Tx, x >< Ry, y > \leq 0$. i.e. $| < Sy, x > |^2 \leq < Tx, x >< Ry, y >$.
($\Leftarrow$) Clear!.

LEMMA 2.3([5]). *Let $T, A \in B(H)$ with $T \geq A^*A$ and $TA = AT$. Then $T \geq AA^*$, and $\begin{pmatrix} A & (T - AA^*)^{1/2} \\ (T - A^*A)^{1/2} & -A^* \end{pmatrix}$ is a normal operator in $B(H \oplus H)$.*

THEOREM 2.4(RUSSO-DYE,[5],[10]). *Let $\phi : U \longrightarrow B$ is a unital positive linear map between two C^*-algebras, then ϕ is a contraction.*

THEOREM 2.5. *Let $\phi : U \longrightarrow B$ is a positive linear map between two C^*-algebras. If T, A are operators in U with $T \geq A^*A$ and $TA = AT$, then $\phi(T) \geq \phi(A^*)\phi(A)$.*

Proof. By lemma 2.3,

$$T \geq AA^* \quad \text{and} \quad N = \begin{pmatrix} A & (T - AA^*)^{1/2} \\ (T - A^*A)^{1/2} & -A \end{pmatrix}$$

is a normal operator in $M_2(U)$. Let $\theta : M_2(U) \longrightarrow U$ be the following map: $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \to A_{11}$. Then $\phi \circ \theta : M_2(U) \longrightarrow B$ is a unital positive map. By the well-knowen *Schwarz inequality*, $\phi \circ \theta(N^*N) \geq \phi \circ \theta(N^*) \cdot \phi \circ \theta(N)$. Since $N^*N = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, $\theta(N^*N) = T$, $\theta(N) = A$. Thus the desired inequality follows.

THEOROM 2.6. *Let $S, T \in B(H)$ with T positive. If* $\begin{pmatrix} T & S \\ S^* & T \end{pmatrix} \geq 0$, *then $S^*S \leq \| T \| \cdot T$ and in particular,* $\| S \| \leq \| T \|$.

Proof. By lemma 2.2, For all x, y in H, $| < Sy, Sy > |^2 \leq < Ty, y > \cdot < Tx, x >$. Let $x = Sy$, then $| < Sy, x > |^2 \leq < Ty, y > \cdot < TSy, Sy > \leq < Ty, y > \cdot \| T \| \cdot \| Sy \|^2$. Thus $< Sy, Sy > \leq \| T \| \cdot < Ty, y >$, and so $S^*S \leq \| T \| \cdot T$.
In particular, $\| S^*S \| = \| S \|^2 \leq \| T \|^2$, and so $\| S \| \leq \| T \|$.

COROLLARY 2.7. *A positive map need not have a positive extension unless the range is the complex field C.*

Proof. Let $S(\Pi)$ be the space of all continuous functions on the unit circle Π and S the subspace spanned by $1, z$ and $\bar{z}$.

Defining $\phi : S(\Pi) \longrightarrow M_2$ by $\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}$, then ϕ is positive and $\| \phi \| = 2 \| \phi(1) \|$ ([5], [10]). If the extension $\tilde{\phi}$ of ϕ is positive on $C(\Pi)$, by theorem 2.4, $\tilde{\phi}$ is contractive. But since $\| \phi \| = 2$, ϕ cannot be contractive.

Remark. In case $T = I$, theorem 2.5 reduces to theorem 2.4. Since normal implies hyponormal in general(the reverse of implication is invalid in the infinity-dimensional case([10])), we can consider whether theorem 2.5 holds for any hyponormal elements.

3. Some characterizations of unital positive map

LEMMA 3.1. *Let $0 < \mu < 1$ and $0 < \epsilon \leq (1/6)\mu^2$. Then the function $p(z_1, z_2, z_3, z_4, t)$ with complex indeterminents given by $p = -\epsilon|z_1|^2 + |z_1 - tz_2|^2 + |z_2 - tz_3|^2 + |z_3 - \bar{t}z_2|^2 + |z_4 - \bar{t}z_3|^2 + |z_3 + \mu t z_1 - tz_4|^2$ is positive semidefinite for every $z_1, z_2, z_3, z_4, t \in C$*

Proof. It is clear by elementary computation, using the *Cauchy inequality* and given conditions.

LEMMA 3.2. *Let $0 < \mu < 1$ and $0 < \epsilon \leq (1/6)\mu^2$. Then the linear map $\phi : M_4 \longrightarrow M_2$ given by* $\phi\Big((a_{ij})_{i,j=1}^4\Big) =$

$$\begin{pmatrix} (1-\epsilon)a_{11} + a_{22} + 2a_{33} + a_{44} & -a_{12} - 2a_{23} + \mu a_{31} - 2a_{34} \\ \mu a_{13} - a_{21} - 2a_{32} - 2a_{43} & \mu^2 a_{11} - \mu a_{14} + 2a_{22} + 2a_{33} - \mu a_{41} + a_{44} \end{pmatrix}$$

is positive.

Proof. It is suffices to prove that $\phi\left(\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}(\bar{s} \quad \bar{t} \quad \bar{u} \quad \bar{v})\right) \geq 0$ $\forall s,t,u,v \in C$.

i.e. $x^\star\phi\left(\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}(\bar{s} \quad \bar{t} \quad \bar{u} \quad \bar{v})\right)x \geq 0$ for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C^2$ and all $s,t,u,v \in C$.

By the way, $(\bar{x}_1, \bar{x}_2)\phi\left(\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}(\bar{s} \quad \bar{t} \quad \bar{u} \quad \bar{v})\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1-\epsilon)|s|^2 x_1\bar{x}_1 + |t|^2 x_1\bar{x}_1 + 2|u|^2 x_1\bar{x}_1 + |v|^2 x_1\bar{x}_1 - s\bar{t}x_2\bar{x}_1 - 2t\bar{u}x_2\bar{x}_1 + \mu u\bar{s}x_2\bar{x}_1 - 2u\bar{v}x_2\bar{x}_1 + \mu s\bar{u}x_1\bar{x}_2 - t\bar{s}x_1\bar{x}_2 - 2u\bar{t}x_1\bar{x}_2 - 2v\bar{u}x_1\bar{x}_2 + \mu^2|s|^2 x_2\bar{x}_2 - \mu s\bar{v}x_2\bar{x}_2 + 2|t|^2 x_2\bar{x}_2 + 2|u|^2 x_2\bar{x}_2 - \mu v\bar{s}x_2\bar{x}_2 + |v|^2 x_2\bar{x}_2$. If $x = 0$, then

$$(\bar{x}_1 \quad \bar{x}_2)\phi\left(\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}(\bar{s} \quad \bar{t} \quad \bar{u} \quad \bar{v})\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = |x_2|^2(\mu^2|s|^2 - \mu s\bar{v} + 2|t|^2 + 2|u|^2 - \mu v\bar{s} + |v|^2) = |x_2|^2(|\mu s - v|^2 + 2|t|^2 + 2|u|^2) \geq 0.$$

If $x_1 \neq 0$, we can assume without loss of generality that $x_1 = 1$. Then

$x^\star\phi\left(\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}(\bar{s} \quad \bar{t} \quad \bar{u} \quad \bar{v})\right)x = -\epsilon|s|^2 + |s|^2 - x_2 s\bar{t} - \bar{x}_2\bar{s}t + |x_2|^2 t\bar{t} + t\bar{t} - x_2 t\bar{u} - \bar{x}_2 u\bar{t} + |x_2|^2 u\bar{u} + u\bar{u} - \bar{x}_2\bar{t}u - x_2 t\bar{u} + |x_2|^2|t|^2 + v\bar{v} - \bar{x}_2\bar{u}v - x_2 u\bar{v} + |x_2|u\bar{u} + u\bar{u} + \mu x_2\bar{s}u - x_2 u\bar{v} + \mu^2|x_2|^2 s\bar{s} - \mu|x_2|^2 s\bar{v} + \mu\bar{x}_2 s\bar{u} - \bar{x}_2\bar{u}v - \mu|x_2|^2\bar{s}v + |x_2|^2 v\bar{v} = -\epsilon|s|^2 + |s - \bar{x}_2 t|^2 + |t - \bar{x}_2 u|^2 + |u - x_2 t|^2 + |u - x_2 t|^2 + |u + \mu\bar{x}_2 s - x_2 v|^2$.

By lemma 3.1, this is positive.

Lemma 3.3. *Let* $x = 1/(1-\mu^2)$ *and* $y = \mu/(1-\mu^2)$.

If $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} x & 0 & 0 & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y & 0 & 0 & x \end{pmatrix}$, *then*

$(a).\quad C-B^{\star}B\geq 0$ and $C-BB^{\star}\geq 0 \qquad (b).\quad \begin{pmatrix}1 & B\\ B^{\star} & C\end{pmatrix}\geq 0.$

Proof. $|(C-B^{\star}B)-\lambda E_4| = \lambda^2(\lambda^2+(1-2x)\lambda+x^2-x-\mu^2x^2)$. Since $y/x=(x-1)/y=\mu$, the eigenvalues of $C-B^{\star}B$ are $0,(1+\mu^2)/(1-\mu^2)$, so nonnegative$(0<\mu<1)$. Since $\begin{pmatrix}1 & B\\ B^{\star}B & C\end{pmatrix} =$ $\begin{pmatrix}I & 0\\ B^{\star} & I\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & C-B^{\star}B\end{pmatrix}\begin{pmatrix}1 & B\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & B\\ B^{\star} & C\end{pmatrix}$ and $\begin{pmatrix}I & 0\\ 0 & C-B^{\star}B\end{pmatrix}$ are congruent. Thus $\begin{pmatrix}1 & B\\ B^{\star} & C\end{pmatrix}\geq 0$.

PROPOSITION 3.4. *Let $0<\mu<1$ and $0<\epsilon\leq(1/6)\mu^2$. Let Ψ : $M_4\longrightarrow M_2$ defined by* $\Psi\Big((a_{ij})_{i,j=1}^4\Big)=\Psi(I)^{-1/2}\Psi\Big((a_{ij})_{i,j=1}^4\Big)\Psi(I)^{-1/2}=$

$$\begin{pmatrix}\alpha\{(1-\epsilon)a_{11}+a_{22}+2a_{33}+a_{44}\} & \beta(-a_{12}-2a_{23}+\mu a_{31}-2a_{34})\\ \beta(\mu a_{13}-a_{21}-2a_{32}-2a_{43}) & \gamma(\mu^2-\mu a_{14}+2a_{22}+2a_{33}-\mu a_{41}+a_{44})\end{pmatrix}$$

,where $\alpha=(5-\epsilon)^{-1}, \beta=\{(5-\epsilon)(5+\mu^2))\}^{-1/2}$, and $\gamma=(5+\mu^2)^{-1}$, and let B, C be as in lemma 3.3. Then Ψ is unital positive and $\Psi(B^{\star})\Psi(B)\not\leq\Psi(C)$.

Proof. By lemma 3.3, $\begin{pmatrix}I & B\\ B^{\star} & C\end{pmatrix}$ and $\begin{pmatrix}I & B^{\star}\\ B & C\end{pmatrix}$ are positive in $M_2(M_4)$. But considering $\begin{pmatrix}5-\epsilon & -5\\ -5 & 5\end{pmatrix}$, one of eigenvalues, $(10-\epsilon)-\sqrt{\epsilon^2+10}/2$ is negative. Thus $\begin{pmatrix}5-\epsilon & -5\\ -5 & 5\end{pmatrix}\not\geq 0$. So

$$\begin{pmatrix}\Psi(I) & \Psi(B)\\ \Psi(B^{\star}) & \Psi(C)\end{pmatrix}=\begin{pmatrix}5-\epsilon & 0 & 0 & -5\\ 0 & 5+\mu^2 & 0 & 0\\ 0 & 0 & * & 0\\ -5 & 0 & 0 & 5\end{pmatrix}\not\geq 0.$$

By lemma 3.2, Ψ is positive and $\Psi(I)=\begin{pmatrix}5-\epsilon & 0\\ 0 & 5+\mu^2\end{pmatrix}$ is positive and invertible. So Ψ is positive and unital. But $\begin{pmatrix}\Psi(I) & \Psi(B)\\ \Psi(B^{\star}) & \Psi(C)\end{pmatrix}=$ $\begin{pmatrix}\Psi(I)^{-1/2} & 0\\ 0 & \Psi(I)^{-1/2}\end{pmatrix}\begin{pmatrix}\Psi(I) & \Psi(B)\\ \Psi(B^{\star}) & \Psi(C)\end{pmatrix}\begin{pmatrix}\Psi(I)^{-1/2} & 0\\ 0 & \Psi(I)^{-1/2}\end{pmatrix}$. Also, by lemma 3.3, $C\geq B^{\star}B$ and $C\geq BB^{\star}$.

Thus $\begin{pmatrix} \Psi(I) & \Psi(B) \\ \Psi(B^\star)\Psi(C) & I \end{pmatrix} \ngeq 0$ and hence $\Psi(C) \ngeq \Psi(B^\star)\Psi(B)$.

COROLLARY 3.5. *Let $B(l^2)$ be the algebra of all bounded linear operator on the Hilbert space l^2 and let $T : B(l^2) \longrightarrow M_2$ defined by $T\Big((a_{ij})^\infty\Big) = \Psi\Big((a_{ij})^4\Big)$. Then there exists a hyponormal element M in $B(l^2)$ such that $T(M^\star M) \ngeq T(M^\star)T(M)$.*

Proof. Let $M = \begin{pmatrix} B & 0 & \cdot & \cdots \\ (C - B^\star B)^{1/2} & 0 & \cdot & \cdots \\ 0 & mI & 0 & \cdots \\ 0 & 0 & m^2 I & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \in B(l^2)$

(since l^2 has a infinity dimension, this is possible).

Then $M^\star M - MM^\star = \begin{pmatrix} C - BB^\star & S & 0 & \cdots \\ S^\star & m^2 I - (C - B^\star B) & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}$

with $S = -B(C - B^\star B)^{1/2}$.

When $m^2 z$ is sufficiently large, since $C - BB^\star$ is invertible, by lemma 2.1,

$M^\star M - MM^\star$ is positive(i.e. M is hyponormal). Next,

$$\begin{aligned} &T(M^\star M) \\ &= T\left(\begin{pmatrix} C & 0 & \cdot & & \\ 0 & m^2 I & 0 & \cdot & \\ 0 & 0 & m^2 I & 0 & \cdot \\ 0 & 0 & 0 & \cdot & \\ & & & & \cdot \end{pmatrix}\right) \\ &= \begin{pmatrix} \alpha\{(1-\epsilon)x + 3 + x\} & 0 \\ 0 & \gamma(\mu^2 x - 2\mu y + 4 + x) \end{pmatrix}. \end{aligned}$$

Also, $T(M^\star)T(M) = \begin{pmatrix} 0 & 0 \\ 0 & 25\beta^2 \end{pmatrix}$. Thus $T(M^\star M) - T(M^\star)T(M) =$

$$\begin{pmatrix} \alpha\{(1-\epsilon)x + 3 + x\} & 0 \\ 0 & \gamma(\mu^2 x - 2\mu y + 4 + x) - 25\beta^2 \end{pmatrix}$$

Considering

$$\begin{aligned}\gamma(\mu^2 x - 2\mu y + 4 + x) - 25\beta^2 &= 5/(5+25\mu^2) - 25/(5-\epsilon)(5+\mu^2)\\ &= -5\epsilon/(5+\mu^2)(5-\epsilon) < 0,\end{aligned}$$

consequently, $T(M^*M) - T(M^*)T(M) \not\geq 0$.

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Department of Mathematics
Changwon National University
Changwon 641-773, Korea