Kangweon–Kyungki Math. Jour. **1** (1993), pp. 93–99

C–TOLERANCE STABILITY OF DYNAMICAL SYSTEMS

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1. Introduction.

Zeeman[8] introduced the concepts of tolerance stability of dynamical systems on a compact metric space. In this paper we investigate tolerance stability of dynamical systems using the concepts of chain recurrence. And we get an equivalence condition for a dynamical system to be tolerance stable. Also we introduce the notion of C–tolerance stability using the concept of chain recurrence and it is shown that an equivalence condition for a dynamical system to be C–tolerance stable. We show that C–tolerance stability is invariant under conjugacy. Finally we give a necessary condition for which the notion of tolerance stability is equal to that of C–tolerance stability.

We consider homeomorphisms (or dynamical systems) acting on a compact metric space. Let X denote a compact metric space with a metric d, and let $H(X)$ be the collection of all homeomorphisms of X to itself topologized by the C^0–metric:

$$d_0(f,g) = \sup\{\ d(f(x), g(x)) \mid x \in X\ \},$$

where f and g are elements in $H(X)$.

We say that a dynamical system $f \in H(X)$ is *topologically stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f,g) < \delta$, $g \in H(X)$, then there is a continuous surjection $h : X \to X$ with $fh = hg$ and $d_0(h, I_X) < \epsilon$, where $I_X : X \to X$ stands for the identity homeomorphism. We introduce the concept of tolerance stability for homeomorphisms which is weaker than that of topologically stability.

Let $K(X)$ be the set of all nonempty closed subsets of X with the Hausdorff metric ρ: for any $A, B \in K(X)$,

$$\rho(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\},$$

where $d(a, B) = \inf\{d(a,b) \mid b \in B\}$. Then the set $K(X)$ with the metric ρ is again a compact metric space. Let $K(K(X))$ be the set of all nonempty closed subsets of $K(X)$ with the Hausdorff metric $\bar{\rho}$.

For any $f \in H(X)$ and $x \in X$, the set

$$O(f,x) = \overline{\{f^n(x) : n \in \mathbf{Z}\}}$$

is called the *f-orbit closure* of x. Since the set $O(f,x)$ can be interpreted as a point in $K(X)$, we can consider the closure of the set $\{O(f,x) : x \in X\}$ in $K(X)$, which is denoted by $O(f)$. The set $O(f)$ also may be interpreted as a point of $K(K(X))$. Hence we can consider the map $O : H(X) \to K(K(X))$ sending $f \in H(X)$ to $O(f)$.

2. Tolerance Stability.

We say that $f \in H(X)$ is *tolerance stable* if the map $O : H(X) \to K(K(X))$, which assigns to each $g \in H(X)$ the point $O(g) \in K(K(X))$, is continuous at f as we know in [1]. Suppose that X and Y are metric spaces with Y compact. A map $h : X \to K(Y)$ is said to be *upper* (or *lower) semi-continuous* at $x \in X$ if for any $\epsilon > 0$ there exists a neighborhood U of x such that for any $z \in U$ we have

$$h(z) \subset B_\epsilon(h(x)) \text{ (or } h(x) \subset B_\epsilon(h(z))\text{)},$$

respectively, where $B_\epsilon(A) = \{z \in X : d(x,z) < \epsilon \text{ for some } x \in A\}$.

A map $h : X \to K(Y)$ is continuous at $x \in X$ if and only if h is upper and lower semicontinuous at $x \in X$. In the following theorem, we see that the notion of tolerance stability is characterized.

THEOREM 2.1. *$f \in H(X)$ is tolerance stable if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f,g) < \delta$ with $g \in H(X)$ then for any $x \in X$ there are $y, z \in X$ satisfying*

$$\rho(O(f,g), O(g,y)) < \epsilon \quad \text{and} \quad \rho(O(f,z), O(g,x)) < \epsilon.$$

Proof. Let $f \in H(X)$ be tolerance stable. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f,g) < \delta$ with $g \in H(X)$, then $O(g) \subset B_{\epsilon/2}(O(f))$ and $O(f) \subset B_{\epsilon/2}(O(g))$. Let $A \in O(g)$. Then there exists

$z \in X$ such that $\rho(A, O(g,z)) < \epsilon/2$. Since $A \in B_{\epsilon/2}(O(f))$, we have $\rho(A, O(f,x)) < \epsilon/2$ for any $x \in X$. Then we have $\rho(O(f,x), O(g,z)) < \epsilon$ for any $g \in B_\delta(f)$. Similarly we can show that if $d_0(f,g) < \delta$ and $x \in X$, then there exists $y \in X$ such that $\rho(O(g,x), O(f,y)) < \epsilon$.

Conversely, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f,g) < \delta$, $g \in H(X)$, then for any $x \in X$ there are $y, z \in X$ satisfying

$$\rho(O(g,y), O(f,x)) \leq \frac{\epsilon}{3} \quad \text{and} \quad \rho(O(f,z), O(g,x)) \leq \frac{\epsilon}{3}.$$

Let $g \in B_\delta(f)$ and $A \in O(g)$. Then there exists $x \in X$ such that $\rho(A, O(g,x)) < \epsilon/2$. For $x \in X$, we select $z \in X$ satisfying

$$\rho(O(g,x), O(f,z)) \leq \frac{\epsilon}{3}.$$

Then we obtain $\rho(A, O(f,z)) < \epsilon$. This means that $O(g) \subset B_\epsilon((O(f))$. Hence the map O is upper semi-continuous at f. Similarly we can show that the map O is lower semi-continuous at f. Consequently the map f is tolerance stable □

Using the concept of chain recurrence, we will introduce the concept of C–tolerance stability of $f \in H(X)$, and show that the notion of C–tolerance stability is characterized. For our purpose we need some notations and definitions(see [7]).

Let x and y be two points in X, and let $\epsilon > 0$ be an arbitrary number. A finite sequence $\{x_i\}_{i=0}^n$ in X is called an ϵ-chain for f from x to y if

(1) $d(x_{i+1}, f(x_i)) < \epsilon$ for $i = 0, 1, \ldots, n-1$ and
(2) $x_0 = x$ and $x_n = y$.

Using the concept, we define a relation "$<$"on X induced by $f \in H(X)$ as follows: for any $x, y \in X$, $x < y$ if and only if for any $\epsilon > 0$ there exists an ϵ-chain for f from x to y. As we know in [3], the f-chain orbit through $x \in X$, $C(f,x) = \{y \in X \mid x < y \text{ or } x > y \text{ or } x = y\}$, is compact in X for each $x \in X$. Since each chain orbit $C(f,x)$ can be interpreted as a point in $K(X)$, we can consider the set $\{C(f,x) \mid x \in X\}$ in $K(X)$, which is denoted by $C(f)$. Then the set $C(f)$ is closed in $K(X)$. Hence the chain orbit map $C : H(X) \to K(K(X))$ sending f to $C(f)$ is well-defind. We say that $f \in H(X)$ is *C–tolerance stable* if the map $C : H(X) \to K(K(X))$ is continuous at f.

THEOREM 2.2. *$f \in H(X)$ is C–tolerance stable if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f,g) < \delta$ with $g \in H(X)$ then for any $x \in X$ there are $y, z \in X$ satisfying*

$$\rho(C(f,x), C(g,y)) < \epsilon \quad \text{and} \quad \rho(C(f,z), C(g,x)) < \epsilon.$$

Proof. Similarly we canshow that the theorem is true by the above theorem. □

COROLLARY 2.3. *If $f \in H(X)$ is tolerance stable then it is C–tolerance stable.*

Proof. It follows immediately from the fact that the orbit closure is contained in the chain orbit. □

We say that $f, g \in H(X)$ are *topologically conjugate* if there exists $h \in H(X)$ such that $hg = fh$. The $h \in H(X)$ is called a *topological conjugacy* between f and g. In the following theorem, we see that C–tolerance stability is invariant under a topological conjugacy.

THEOREM 2.4. *Any homeomorphism which is topologically conjugate to a C–tolerance stable homeomorphism is also C–tolerance stable.*

Proof. Let $f \in H(X)$ be C–tolerance stable, and suppose that $f, g \in H(X)$ are topologically conjugate. Let $h \in H(X)$ be a topological conjugacy between f and g. Let $\epsilon > 0$ be arbitrary and choose $0 < \epsilon_1 < \epsilon$ such that if $d(a,b) < \epsilon_1$ then $d(h^{-1}(a), h^{-1}(b)) < \epsilon$ for $a, b \in X$. Applying Theorem 2.2, we shall complete the proof by showing that g is C–tolerance stable. Since f is C–tolerance stable, given $\epsilon_1 > 0$, there exists $\delta > 0$ such that if $d_0(f, f_0) < \delta$ then for any $x \in X$ there are $y, z \in X$ satisfying

$$\rho(C(f_0,y), C(f,x)) < \epsilon_1 \quad \text{and } \rho(C(f,z), C(f_0,x))\epsilon_1.$$

For the $\delta > 0$, choose $0 < \delta_1 < \delta$ such that if $d(a,b) < \delta_1$, $a, b \in X$, then $d(h(a), h(b)) < \delta$. Let $g_0 \in H(X)$ be such that $d_0(g, g_0) < \delta_1$, and let $f_0 = hg_0h^{-1}$. Then we have

$$d(h(g(x)), h(g_0(x))) = d(f(h(x)), f_0(h(x))) < \delta$$

for any $x \in X$, and so $d_0(f, f_0) < \delta$. Then for any $x \in X$, there exists $h(y) \in X$ such that

$$\rho(C(f_0, h(y)), C(f, h(x))) < \epsilon_1.$$

Hence we have

$$\begin{aligned}\rho(C(f_0, h(y)), C(f, h(x))) &= \rho(C(hg_0h^{-1}, h(y)), C(hgh^{-1}, h(x))) \\ &= \rho(C(hg_0, y), C(hg, x)) \\ &< \epsilon_1.\end{aligned}$$

This means that given $\epsilon > 0$, there exists $\delta_1 > 0$ such that, for every $x \in X$ there is $y \in X$ satisfying

$$\rho(C(g_0, y), C(g, x)) < \epsilon.$$

Simiarly we can show that if $d_0(g, g_0) < \delta_1$, $g_0 \in H(X)$, then for any $x \in X$ there exists $z \in X$ satisfying

$$\rho(C(g, z), C(g_0, x)) < \epsilon.$$

This completes the proof. □

Finally we give a necessary condition to be $C(f) = O(f)$, where $f \in H(X)$. For this object, we need a lemma due to Z. Nitecki and M. Shub[5].

LEMMA 2.5. *Let M be a compact manifold of dim≥ 2 with the metric d, and let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that if $\{(x_i, y_i) \in M \times M \mid i = 1, 2, \ldots, n\}$ is a finite set of points of $M \times M$ satisfying*

(1) *for each $i = 1, 2, \ldots, n$, $d(x_i, y_i) < \delta$ and*
(2) *if $i \neq j$, then $x_i \neq x_j$ and $y_i \neq y_j$,*

then there exists $h \in H(M)$ with $d_0(h, 1_M) < \epsilon$ and $h(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

THEOREM 2.6. *Let M be a compact manifold of dim≥ 2. If $f \in H(M)$ is topologically stable, then we have $C(f) = O(f)$.*

Proof. By definition, it is clear that $O(f) \subset C(f)$. Thus it is enough to show that $C(f,x) \subset O(f,x)$ for any $x \in M$. Let d be the metric on M, and let $y \in C(f,x)$. Then we have $x < y$, or $x > y$, or $x = y$. Suppose that $x < y$, and let $k > 0$ be a positive integer. Since f is topologically stable, given $1/k > 0$, there exists $\delta_1(k) > 0$ such that if $d_0(f,g) < \delta_1$ with $g \in H(M)$, then there is a continuous surjection $h : M \to M$ with $fh = hg$ and $d_0(h, I_M) < 1/k$. Given $\frac{1}{k} > 0$, we choose $\delta_2(k) > 0$ satisfying the results of Lemma 2.5. Let $\{x_0, x_1, \ldots, x_{m_k}\}$ be a δ_2-chain for f from x to y. Then the set $\{(f(x_0), x_1), \ldots, (f(x_{m_k-1}, x_{m_k})\}$ satisfies the hypothesis of Lemma 2.5. Hence there exists $\varphi \in H(M)$ such that

$$d_0(\varphi, I_M) < \frac{1}{k} \quad \text{and } \varphi(f(x_i)) = x_{i+1}$$

for $i = 0, 1, \ldots, m_k - 1$. By letting $g = \varphi f$, we get $d_0(f,g) < \delta_1$. Thus there is a continuous surjection h with $fh = hg$, and we get

$$d(f^{m_k}(x), y) = d(f^{m_k}(x), g^{m_k}(x)) < \frac{m_k}{k}.$$

This implies that $B_\epsilon(y) \cap O(f,x) \neq \emptyset$ for any $\epsilon > 0$, and so $y \in O(f,x)$. By now we have shown that if $x < y$ then $y \in O(f,x)$. Similarly we can show that if $x > y$ then $y \in O(f,x)$. This completes the proof. □

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