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ON FUZZY SUPRA SPACES

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§ 1. Introduction and Preliminaries

Fuzzy topological spaces were first introduced in the iterature by Chang [1] who studied a number of the basic concepts such as open set, closed set, neighborhood, interior set and continuity.

In this paper we introduce fuzzy supra topological spaces and fuzzy fuzzy supra neighborhood, fuzzy suprainterior sets and fuzzy supra continuous.

Thoughout this paper, the symbol I will denote the unit interval. Let X be a non-empty set.

DEFINITION 1.1. *A fuzzy set in X is a function with domain X and values in I, that is, an element of I^X.*

DEFINITION 1.2. *Let $\mu, \nu \in I^X$, we define the following fuzzy sets:*

(i) $\mu \wedge \nu \in I^X$, by $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ for each $x \in X$.
(ii) $\mu \vee \nu \in I^X$, by $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ for each $x \in X$.
(iii) $\mu^c \in I^X$, by $\mu^c(x) = 1 - \mu(x)$ for each $x \in X$.

The symbol 0 will be used to denote an empty fuzzy set ($0(x) = 0$ for all $x \in X$). For 1, we have by definition $1(x) = 1$ for all $x \in X$.

DEFINITION 1.3. *Let $T \subset I^X$ satisfy the following conditions:*

(i) $0, 1 \in T$
(ii) If $\mu_1, \mu_2 \in T$, then $\mu_1 \wedge \mu_2 \in T$
(iii) If$\{\mu_j : j \in J\} \in T, V_{j \in J}\{\mu_j\} \in T$.

T is called a fuzzy topology on X and (X, T) a fuzzy topological space (or f.t.s for brevity). The elements of T are called fuzzy open sets.

§ 2. Fuzzy supra topological spaces.

DEFINITION 2.1. *Let $T^* \subset I^X$ satisfy the following conditions :*

(i) $1 \in T^*$
(ii) If $\mu_i \in T^*, i \in I$, then $V_{i \in I}\mu_i \in T^*$

T^* is called a fuzzy supra topological space (or f.s.t.s for brevity). The elements of T^* are called fuzzy supra open sets. And a fuzzy set μ is supra closed iff $1-\mu$ is a supra open.

DEFINITION 2.2. *A fuzzy set μ in a f.s.t.s (X, T^*) is a supra neighborhood, or supra-nbhd for short, of a fuzzy ν if and only if there exists a fuzzy supra open set ω such that $\nu \leq \omega \leq \mu$.*

THEOREM 2.1. *A fuzzy set μ is supra open iff for each fuzzy set ν contains in μ, μ is a supra nbhd of ν.*

proof. ($\Longrightarrow$) Obvious
($\Longleftarrow$) Since $\mu \leq \mu$, there exists a fuzzy supra open ν such that $\mu \leq \nu \leq \mu$. Therefore $\mu = \nu$ and μ is supera open.

DEFINITION 2.3. *Let μ and ν be a fuzzy sets in a f.s.t.s (X, T^*), and let $\mu \geq \nu$. Then ν is called supra interior fuzzy set of μ if and only if μ is a supra-nbhd of ν. The union of all supra interior fuzzy sets of μ is called the supra interior of μ and is denoted by (μ^{si}).*

THEOREM 2.2. *Let μ be a fuzzy set in a f.s.t.s (X, T^*). Then μ^{si} is supra open and is the largest supra-open fuzzy set contained in μ. The fuzzy set μ is supra open iff $\mu = \mu^{si}$.*

proof. Clearly, $\mu^{si}$is itself a supra interior fuzzy fuzzy set of μ. Hence there exists a supra open fuzzy set ν such that $\mu^{si} \leq \nu \leq \mu$. But ν is a supra interior fuzzy set of μ. Hence $\mu^{si} = \nu$. Therefore μ^{si} is supra open and is the largest supra open fuzzy set contained in μ.

If fuzzy set μ is supra open, then $\mu \leq \mu^{si}$. For μ^{si} is a supra interior fuzzy set of μ, $\mu = \mu^{si}$.

The converse is obviously true.

DEFINITION 2.4. *Let μ be a fuzzy set in a f.s.t.s (X, T^*). Then $\wedge\{$ supra closed fuzzy set ν in $X : \nu \geq \mu\}$ is called the supra closure of μ and is denoted by μ^{scl}.*

We obtain the following result :

THEOREM 2.3. *Let μ be a fuzzy set in a f.s.t.s (X, T^*). Then μ^{scl} is supra closed.*

proof. Since $1-\mu^{scl} = 1-\wedge\{\nu|1-\nu \in T^*, \nu \geq \mu\} = \vee\{1-\nu|1-\nu \in T^*, \nu \geq \mu\}$, $1-\mu^{scl}$ is a supra open. Therefore μ^{scl} is supra closed.

THEOREM 2.4. *Let μ and ν be fuzzy sets in a f.s.t.s (X, T^*). If $\mu \leq \nu, \mu^{scl} \leq \nu^{scl}$ and $\mu^{si} \leq \nu^{si}$. Also, $(\mu^{scl})^{scl} = \mu^{scl}$ and $(\mu^{si})^{si} = \mu^{si}$*

proof. These results follow from the approciate definitions.

THEOREM 2.5. *Let μ and ν be fuzzy sets in f.s.t.s (X, T^*)*

(i) $(\mu^{scl} \wedge \nu^{scl}) \geq (\mu \wedge \nu)^{scl}$
(ii) $(\mu \vee \nu)^{si} \geq \mu^{si} \vee \nu^{si}$
(iii) $(1 - \nu)^{si} = 1 - \nu^{scl}$
(iv) $1 - \mu^{si} = (1 - \mu)^{scl}$.

proof. By Theorem 2.4, the proof of (i) and (ii) are clear.

(iii) $1 - \mu^{scl} = 1 - \wedge\{\nu : 1 - \nu \in T^*$ and $\nu \geq \mu\} = \vee\{1 - \nu : 1 - \nu \in T^*$ and $\nu \geq \mu\} = \vee\{\omega : \omega \in T^*$ and $\omega \leq 1 - \mu\} = (1 - \mu)^{si}$, where $\omega = 1 - \nu$.

The proof of (iv) is similar.

THEOREM 2.6. *Let μ and ν be a fuzzy sets in f.s.t.s (X, T^*). Then*

(i) $\mu^{scl} \vee \nu^{scl} < (\mu \vee \nu)^{scl}$
(ii) $(\mu \wedge \nu)^{si} < \mu^{si} \wedge \nu^{si}$.

It can be easily shown, by examples that the inequalities in (i) and (ii) cannot be replaced, in general, by equalities as in case of fuzzy topological spaces.

Ex 1. Let $X = I$ and fuzzy subsets of X are defined as

$$\mu_1(x) = \begin{cases} x, 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}, \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 - x, 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{4}, \frac{1}{2} < x \leq 1, \end{cases}$$

$$\mu_3(x) = \begin{cases} \frac{1}{4}, 0 \leq x \leq \frac{1}{2}, \\ x, \frac{1}{2} < x \leq 1, \end{cases}$$

$$\mu_4(x) = \begin{cases} \frac{1}{2}, 0 \leq x \leq \frac{1}{2}, \\ 1 - x, \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\mu_5(x) = \begin{cases} 1 - x, 0 \leq x \leq \frac{1}{2}, \\ x, \frac{1}{2} \leq x \leq 1, \end{cases}$$

Let $T^* = \{0, \mu_1 \vee \mu_2, \mu_3 \vee \mu_4, \mu_5, 1\}$ be fuzzy supra topology on X.

Since $(\mu_1 \vee \mu_2)^c = \mu_1$ and $(\mu_3 \vee \mu_4)^c = \mu_4, \mu_1$ and μ_4 are supra closed.

Since $\mu_1^{scl} \vee \mu_4^{scl} = \mu_1 \vee \mu_4 \neq 1$ and $(\mu_1 \vee \mu_4)^{scl} = 1, \mu_1^{scl} \vee \mu_4^{scl} < (\mu_1 \vee \mu_4)^{scl}$

(ii) Since $((\mu_1 \vee \mu_2) \wedge (\mu_3 \vee \mu_4))^{si} = 0$ and $(\mu_1 \vee \mu_2)^{si} \wedge (\mu_3 \vee \mu_4)^{si} = (\mu_1 \vee \mu_2) \wedge (\mu_3 \vee \mu_4) \neq 0, ((\mu_1 \vee \mu_2) \wedge (\mu_3 \vee \mu_4))^{si} < (\mu_1 \vee \mu_2)^{si} \wedge (\mu_3 \vee \mu_4)^{si}$.

DEFINITION 2.5. *A function f from a f.s.t.s (X, T^*) to a f.s.t.s (Y, U^*) is FS^*-continuous iff the inverse of each $\mu \in U^*$ is $f^{-1}(\mu) \in T^*$.*

THEOREM 2.7. *If X and Y are f.s.t.s's, and f is a function on (X, T^*) to (Y, U^*), then following conditions are equivalent:*

(1) *The function f is FS^*-continuous.*
(2) *The inverse image of every fuzzy supra closed set is fuzzy supra closed set.*
(3) *For every fuzzy set μ in X, $f(\mu^{scl}) \leq f(\mu)^{scl}$.*
(4) *For every fuzy set ν in Y, $f^{-1}(\nu)^{scl} \leq f^{-1}(\nu^{scl})$.*

proof. (1)$\Longleftrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (3). Now $f(\mu)^{scl} = \wedge\{\nu : \nu \geq f(\mu)$ and $1 - \nu \in U^*\}$. And $f^{-1}(f(\mu)^{scl}) = \wedge\{f^{-1}(\nu) : \nu \geq f(\mu)$ and $1-\nu \in U^*\}$. Since $f^{-1}(\nu)$ is a supra closed fuzzy set and $f^{-1}(\nu) \geq \mu^{scl}$. Therefore $\wedge\{f^{-1}(\nu)\} \geq \mu^{scl}$. Thus $f^{-1}(f(\mu)^{scl}) \geq \mu^{scl}$. So, $f(\mu)^{scl} \leq f(\mu)^{scl}$.

(3) $\Longrightarrow$ (4). Since $f^{-1}(\nu)$ is a fuzzy set in X, it follows that $f(f^{-1}(\nu)^{scl}) \leq f(f^{-1}(\nu))^{scl} \leq \nu^{scl}$. Hence $f^{-1}(\nu)^{scl} \leq f^{-1}(\nu^{scl})$.

(4) $\Longrightarrow$ (2). Let ν be a supra closed fuzzy set in (Y, U^*). Then $f^{-1}(\nu)^{scl} = f^{-1}(\nu^{scl}) = f^{-1}(\nu)$. Therefore $f^{-1}(\nu)$ is a supra closed set in (X, T^*).

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