

COHEN-MACAULAY PROPERTY OF GRADED RINGS ASSOCIATED TO EQUIMULTIPLE IDEALS

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1. Introduction.

The purpose of this paper is to extend C.Huneke and J.Sally's result ([2], Proposition 3.3) to the case of an equimultiple ideal.

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring (R, m) , we mean a Noetherian ring R which has a unique maximal ideal m . By $\dim(R)$ we always mean the Krull dimension of R . Let (R, m) be a local ring and I an ideal of R . An ideal J contained in I is called a reduction of I if $JI^n = I^{n+1}$ for some integer $n \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I .

The reduction number of I with respect to J is defined by

$$r_J(I) = \min \{ n \geq 0 \mid JI^n = I^{n+1} \}.$$

The reduction number of I is defined by

$$r(I) = \min \{ r_J(I) \mid J \text{ is a minimal reduction of } I \}.$$

The analytic spread of I , denoted by $l(I)$, is defined to be $\dim(\overline{R[It]}/mR[It])$. In [5], it is shown that $ht(I) \leq l(I) \leq \dim(R)$. An ideal I is called equimultiple if $l(I) = ht(I)$. If R/m is an infinite field, then $l(I)$ is the least number of elements generating a reduction of I ([5]). In particular, all m -primary ideals are equimultiple. We will use notation $\lambda_R(M)$ (or simply $\lambda(M)$) to denote the length of M as an R -module, $\mu(I)$ to denote the number of elements in a minimal basis of an ideal I in R (i.e., $\mu(I) = \lambda(I/mI)$). As a general reference, we refer the reader to [4] for any unexplained notation or terminology.

2. Preliminaries.

In a d -dimensional local ring (R, m) , for an m -primary ideal I , the Hilbert function $H_I(n) = \lambda(R/I^n)$ is a polynomial in n of degree d with leading coefficient $e(I)/d!$ for $n \geq 0$, i.e.,

$$e(I) = \lim_{n \rightarrow \infty} \frac{d!}{n^d} \lambda(R/I^n).$$

The integer $e(I)$ is called the multiplicity of I , and $e(R)$ means the multiplicity of the maximal ideal m of R .

THEOREM 2.1. ([1], (1)) If (R, m) is a Cohen-Macaulay local ring such that R/m is infinite, then $\mu(m) \leq e(R) + \dim(R) - 1$.

DEFINITION 2.2. A Cohen-Macaulay local ring (R, m) is said to have minimal multiplicity if $\mu(m) = e(R) + \dim(R) - 1$.

LEMMA 2.3. If a d -dimensional local ring (R, m) with infinite residue field satisfies the equation of minimal multiplicity, i.e., $\mu(m) = e(R) + d - 1$, then R is Cohen-Macaulay if and only if for some minimal reduction J of m , $Jm = m^2$.

Proof : If $Jm = m^2$, then

$$\begin{aligned} e(J) &= e(R) \\ &= \mu(m) - d + 1 \\ &= \lambda(m/m^2) - \lambda(J/mJ) + 1 \\ &= \lambda(m/Jm) - \lambda(J/mJ) + 1 \\ &= \lambda(m/J) + \lambda(R/m) \\ &= \lambda(R/J). \end{aligned}$$

Hence R is Cohen-Macaulay by Theorem 17.11 of [4]. Conversely, if R is Cohen-Macaulay, then the assertion is clear by Theorem 1 in [6]. \square

LEMMA 2.4. ([3], Lemma 3) Let a_1, \dots, a_n be elements of a ring A constituting an R -sequence in any order. Let J be an ideal generated by monomials in a_2, \dots, a_n . Then $ba_1 \in J$ implies $b \in J$.

3. Main result.

THEOREM 3.1. Let (R, m) be a d -dimensional Cohen-Macaulay local ring with an infinite residue field R/m . Assume that I is an equimultiple ideal of R with $ht(I) = s > 0$ satisfying $I^2 = (a_1, \dots, a_s)I$ for some minimal reduction a_1, \dots, a_s of I . Then $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal.

Proof : Let $r = d - s$. Since $\dim(R/I) = \dim(R) - ht(I)$, we can choose elements $b_1, \dots, b_r \in m$ such that the images of the b_i in R/I are a system of parameters for R/I . Then $(a_1, \dots, a_s, b_1, \dots, b_r)$ is a system of parameters for R . By Corollary 2.7 in [8], the images a_1^*, \dots, a_s^* in I/I^2 form a regular sequence on $gr_I(R)$. Let $T = gr_I(R)/mgr_I(R)$ and $\dim(T) = l(I) = s$ by the definition of analytic spread.

Claim 1 : a_1' is a regular element on T .

We proceed by induction on $s = ht(I)$. We have two cases : (1) when $s = 1$, and (2) when $s > 1$.

Case(1) : $s = 1$. We have $I^2 = (a_1)I$. Suppose that a_1' is not a regular element on T . Then there exists a non-zero element $c' = c + mI^l$ in T such that $a_1'c' = 0$. Hence we have

$$a_1c \in mI^{l+1} = m(a_1)I^l.$$

Since a_1 is a regular element on R , $c \in mI^l$, which is a contradiction.

Case(2) : $s > 1$. We have $I^2 = (a_1, \dots, a_s)I$. Suppose that a_1' is not a regular element on T . Then there exists a non-zero element $c' = c + mI^l$ in T such that $a_1'c' = 0$. Hence we have

$$a_1c \in mI^{l+1} = m(a_1, \dots, a_s)I^l = mI(a_1, \dots, a_s)^l.$$

Express a_1c as a homogeneous polynomial of degree l in a_1, \dots, a_s with coefficients in mI . Hence we have $a_1(c - h(a_1, \dots, a_s)) = g(a_2, \dots, a_s)$, where $h \in mI[a_1, \dots, a_s]$ is a homogeneous polynomial of degree $l - 1$ in a_1, \dots, a_s with coefficients in mI and $g \in mI[a_2, \dots, a_s]$ is a homogeneous polynomial of degree l in a_2, \dots, a_s with coefficients in mI , i.e.,

$$a_1(c - h(a_1, \dots, a_s)) \in mI(a_2, \dots, a_s)^l \subseteq (a_2, \dots, a_s)^l.$$

By Lemma 2.4, we have $c - h(a_1, \dots, a_s) \in (a_2, \dots, a_s)^l$. Express $c - h(a_1, \dots, a_s) = f(a_2, \dots, a_s)$, where $f \in R[a_2, \dots, a_s]$ is a homogeneous polynomial of degree l in a_2, \dots, a_s with coefficients in R . Moreover,

$$\begin{aligned} a_1 f(a_2, \dots, a_s) &= a_1(c - h(a_1, \dots, a_s)) \\ &\in mI(a_2, \dots, a_s) \\ &\subseteq mI^{l+1}. \end{aligned}$$

By analytic independence, all the coefficients of f are in m . Hence we have

$$\begin{aligned} c &= h(a_1, \dots, a_s) + f(a_2, \dots, a_s) \\ &\in mI(a_1, \dots, a_s)^{l+1} + mI^l \\ &\subseteq mI^l, \end{aligned}$$

which is a contradiction. This finishes the proof of claim 1. Pass to the Cohen-Macaulay local ring $R_1 = R/a_1R$. Then $I_1 = IR_1$ is an equimultiple ideal of $ht(I_1) = s - 1$ and $I_1(a_2, \dots, a_s)R_1 = I_1^2$. The induction hypothesis applies to R_1 , so a'_2, \dots, a'_s form a regular sequence in $gr_{I_1}(R_1)/m_1gr_{I_1}(R_1)$ and hence $gr_{I_1}(R_1)/m_1gr_{I_1}(R_1)$ is Cohen-Macaulay, where $m_1 = mR_1$. But, since a_1^* is a regular element in $gr_I(R)$, we have the following isomorphism:

$$\frac{gr_{I_1}(R_1)}{m_1gr_{I_1}(R_1)} \cong \frac{gr_I(R)/mgr_I(R)}{a_1^*(gr_I(R)/mgr_I(R))}.$$

Hence $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay since a_1^* is a regular element in T by claim 1. Let $N = I/mI \oplus I^2/mI^2 \oplus \dots$ be the unique maximal homogeneous ideal of T . Hence T_N is a Cohen-Macaulay local ring of $\dim(T_N) = s$.

Claim 2 : T_N satisfies the equation of minimal multiplicity. Since $a'_1, \dots, a'_s \in I/mI$, $(a'_1, \dots, a'_s) \subseteq N$. Hence

$$\begin{aligned} (a'_1, \dots, a'_s)N &= \frac{(a_1, \dots, a_s)I}{mI^2} \oplus \frac{(a_1, \dots, a_s)I^2}{mI^3} \oplus \dots \\ &= I^2/mI^2 \oplus I^3/mI^3 \oplus \dots \\ &= N^2. \end{aligned}$$

Therefore

$$\begin{aligned}
1 + \mu(NT_N) &= \lambda(T_N/NT_N) + \lambda(NT_N/N^2T_N) \\
&= \lambda(T_N/N^2T_N) \\
&= \lambda(T_N/(a'_1, \dots, a'_s)T_N) + \lambda((a'_1, \dots, a'_s)T_N/N^2T_N) \\
&= \lambda(T_N/(a'_1, \dots, a'_s)T_N) \\
&\quad + \lambda((a'_1, \dots, a'_s)T_N/(a'_1, \dots, a'_s)T_NNT_N) \\
&= e(T_N) + \mu((a'_1, \dots, a'_s)T_N) \\
&= e(T_N) + s \\
&= e(T_N) + \dim(T_N). \quad \square
\end{aligned}$$

COROLLARY 3.2. ([2], Proposition 3.3.) Let (R, m) be a d -dimensional Cohen-Macaulay local ring and I an m -primary ideal satisfying $I^2 = (a_1, \dots, a_d)I$ for some minimal reduction a_1, \dots, a_d of I . Then $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal.

Proof : Any m -primary ideal in a local ring (R, m) is an equimultiple ideal. \square

After this paper was completed, I learned that K.Shah proved independently the result in this title. But I used different technics to prove the main result.

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